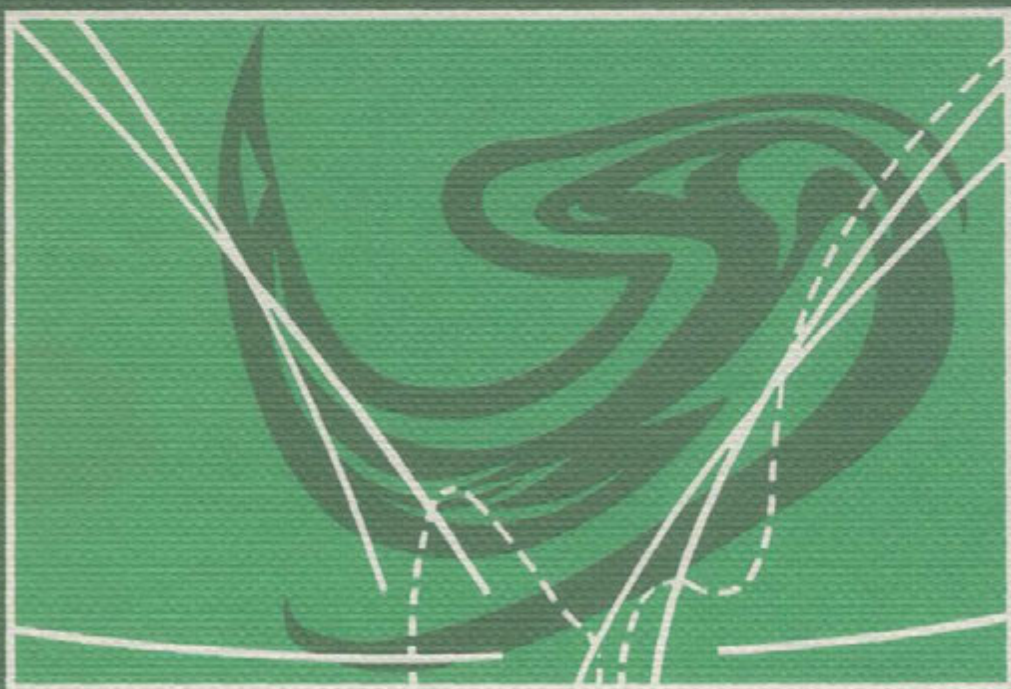


BIFURCATION AND CHAOS IN SIMPLE DYNAMICAL SYSTEMS

J. Awrejcewicz



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PREFACE

This book consists of three chapters. In the first chapter, analytical methods to solve the Hopf bifurcation problem are presented. These methods are based on the perturbation and harmonic balance techniques well known in nonlinear dynamics. One parameter Hopf bifurcation, biparameter Hopf bifurcation and bifurcation into quasiperiodic orbits in autonomous systems are considered. Then the analytical approach for detecting Hopf bifurcation solutions in Duffing, Mathieu and Mathieu-Duffing oscillators are demonstrated.

The second chapter includes a numerical approach to systematically study the behaviour of mechanical and bio-mechanical systems governed by the non-autonomous and autonomous nonlinear differential equations. The procedure presented is based on solving a boundary value problem using the shooting method. The behaviour of the dynamical systems are traced along variations of the chosen parameters. Observation of the evolution of the characteristic multipliers gives information about stability and possibly further branching of the investigated solutions. In the case of nonautonomous systems, periodic orbits are traced and the transition to chaos when one of the multipliers crosses the unit circle at $+1$ or -1 (period doubling) are discussed and illustrated. An example is given for chaos, which has appeared after the simultaneous passage of a pair of multipliers through the unit circle of the complex plane. An autonomous system of three nonlinear differential equations which governs the oscillations of human vocal cords is considered in some detail. Steady state solutions, Hopf bifurcations and the branches of periodic orbits which emanate from bifurcation points are calculated. Tracing the evolution of characteristic multipliers allows the observation of alternations of stability and possibly branching to other periodic or quasiperiodic motion.

In Chapter 3, the attention concentrates on formulating the analytical condition for Hopf bifurcation of the periodic orbit or the Hopf type bifurcation of the quasiperiodic orbit which lead to nonlinear algebraic bifurcation equations. Three examples show the occurrence of chaotic orbits after bifurcations.

The contents of this book are considerably influenced by my experience at the seminar at the Polish Academy of Sciences in Warsaw (1984–1987) and at the seminar organized by the Stefan Banach International Mathematical Center (September 19–December 18, 1986). I express my thanks to Prof. W. Szemplińska-Stupnicka for his knowledgeable comments on part of the results presented in Chapters 1 and 3.

A majority of results presented in this book are based on a research project supported by the Alexander von Humboldt Foundation. I would like to acknowledge the many useful discussions with Prof. E. Brommundt, Prof. D. Ottl and H. Staben regarding the material given in Chapter 2. I wish to express my sincere thanks for

the hospitality of the Institute of Technical Mechanics of the Carolo Wilhelmina University of Braunschweig, where part of the work presented in this book was carried out.

Finally I acknowledge a very fruitful collaboration with R. Klopp from the University of Waterloo, who has helped avoid many difficulties with the style and presentation of my work.

JAN AWREJCEWICZ

Braunschweig, Spring 1989

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Chapter 1

1. HOPF BIFURCATION PROBLEM: AN ANALYTICAL APPROACH

1.1. Introduction

Consider the dynamical system governed by the system of ordinary differential equations

$$\dot{x} = F(\eta, x) = F_{\eta}(x) \quad (1)$$

with $x \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^k$. Suppose, that F is smooth and dependent upon parameter η . Let, for η_0 , system (1) possesses the constant solution $x(\eta_0) = x_0\eta_0$. Generally, system (1) has $n+2$ topologically different portraits of the phase flow. They correspond to one nonsingular point and $n+1$ simple singular points $x_0^{p,q}$, where $p = 0, 1, \dots, n$, $p+q = n$ (see [1]). Numbers p and q are the numbers of the eigenvalues of the Jacobian matrix $DF_x(\eta_0, x_0)$ with p negative and q positive real parts. Phase trajectories near the nonsingular point are similar to the parallel lines. The behaviour of the phase trajectories in the neighborhood of the singular point $x_0^{p,q}$ depend on p and q . For $p = n$ all trajectories aspire to the stable point $x_0^{n,0}$ when $t \rightarrow +\infty$, whereas for $q = n$ all the trajectories aspire to the unstable point $x_0^{0,q}$ when $t \rightarrow -\infty$. All the other points $x_0^{p,q}$ are saddle points. Two surfaces S_p^+ and S_q^- with codimension p and q , respectively pass through the saddle point. If the trajectory belongs to the surface S_p^+ (S_q^-) the point $x_0^{p,q}$ is the stable one $x_0^{p,0}$ (unstable one $x_0^{0,q}$). The stable (unstable) singular point corresponds to the stable (unstable) equilibrium point of the considered dynamical system. By continuously changing the parameter η the numbers p and q change. This change takes place if one (or a pair) of eigenvalues of the characteristic equation

$\det \| DF_x(x_0) - \lambda I \| = 0$ cross the imaginary axis of the complex plane λ . This means, that a bifurcation from an equilibrium state appears either when a simple zero eigenvalue or a simple pair of pure imaginary eigenvalues passes through the imaginary axis. The latter case known as a Hopf bifurcation is investigated in this chapter.

The Hopf bifurcation is up to now widely described in literature. The extensive list of references connecting with this question is for instance given by Marsden and McCracken [2], Iooss and Joseph [3], Hassard, Kazarinoff and Wan [4]. The aim of this chapter is to present applications of approximate analytical methods to obtain the new bifurcated solutions (i. e. perturbation technique and harmonic balance method). Mentioned approximate analytical methods are well documented (see for instance [5-9]). In this chapter three different methods are presented to solve the Hopf bifurcation problem. In the first considered example $\eta \in \mathbb{R}^1$ and one pair of purely imaginary eigenvalues cross the imaginary axis. The use of the center manifold theorem [10] allows the approximate solution, which exactness is improved by the method of successive approximations, to be easily obtained. In the second example with $\eta \in \mathbb{R}^2$ the new periodic solution is sought in the form of a power series in relation to parameter η and another parameter connected with the amplitude of oscillations. The third example presents the method to obtain quasiperiodic solutions which appear after two pairs of purely imaginary eigenvalues cross the imaginary axis. In this case two independent perturbation parameters ξ_1 and ξ_2 (connected with the amplitudes of oscillations) are involved in order to obtain postbifurcation solutions. The Hopf bifurcation problem in autonomous systems is widely investigated by Huseyin and coworkers (see for instance [11-14]).

The investigation of external periodic forcing and parametric periodic perturbations on the Hopf bifurcating solutions are more complicated when compared with the autonomous systems [15-18] and is also considered here.

All the results of this chapter are based on the results given in [19-24] (see also [25,26]).

1.2. One Parameter Hopf Bifurcation

Consider the differential equation (1) where $\eta \in \mathbb{R}^1$. Suppose, that the characteristic equation which corresponds to the Jacobian matrix $DF_\eta(x_0, \eta)$ has the form

$$(\delta - \delta_1(\eta))(\delta - \delta_2(\eta)) P(\eta, \delta) = 0, \quad (2)$$

where

$$\begin{aligned} \delta_{1,2} &= \xi(\eta) \pm i\omega(\eta), \\ \omega(\eta_c) &\neq 0, \\ \xi(\eta_c) &= 0, \\ \xi'(\eta_c) &\neq 0, \end{aligned} \quad (3)$$

and η_c is the critical value of parameter η . Assume that equation (1) can be presented in the form

$$\begin{aligned} \dot{u} &= K_u(\eta)u + K_v(\eta)v + \tilde{K}(\eta, u, v), \\ \dot{v} &= S_u(\eta)u + S_v(\eta)v + \tilde{S}(\eta, u, v), \end{aligned} \quad (4)$$

where $u \in \mathbb{R}^2$, $v \in \mathbb{R}^{n-2}$, $\eta \in \mathbb{R}^1$, $x = \text{col}[u, v]$ and matrixes $K_u(\eta)$, $K_v(\eta)$, $S_u(\eta)$, $S_v(\eta)$ are the linear parts of expansion of $F(x, \eta)$ into the Taylor series around the equilibrium point x_0 ($x_0 = 0$ is taken into further consideration). Equation $P(\eta, \delta) = 0$ has roots with negative real parts, for η in the neighbourhood of η_c , and δ_1 and δ_2 are eigenvalues of the two dimensional matrix $K_u(\eta_c)$. The matrix K_u , known also as the critical matrix, determines the Hopf bifurcation.

From the center manifold theorem, in the vicinity of the equilibrium path $x_0 = 0$, there exists a function, $v = f(u)$

which has the property $f_u = 0^\dagger$. For the first approximation, one can assume that

$$v = f(u) = 0. \quad (5)$$

Taking (5) in (4) into account one can obtain

$$\dot{u} = K_u(\eta)u + \tilde{K}(\eta, v), \quad (6)$$

where $\tilde{K}(\eta, u) = \tilde{K}(\eta, u, 0)$. In the neighbourhood of the critical point

$$K_u(\eta) = K_u(\eta_c) + K_{u\eta}(\eta - \eta_c) + \frac{1}{2}K_{u\eta\eta}(\eta - \eta_c)^2 + \dots \quad (7)$$

Let $u(\eta) = 0$ be the solution of (6) for $\eta < \eta_c$ and at the point $\eta = \eta_c$ the periodic solution $u(t, \varepsilon) = u(t + T, \varepsilon)$ of the period T bifurcates from equilibrium. This solution depends on the formally assumed perturbation parameter connected with the amplitude of oscillations. Then dimensionless time $\tau = \omega t$ is introduced and from (6) one obtains

$$\omega(\varepsilon) u_\tau(\tau, \varepsilon) = K(\eta, \tau, \varepsilon), \quad (8)$$

where $K(\eta, \tau, \varepsilon) = K_u(\eta)u(\tau, \varepsilon) + \tilde{K}(\eta, u(\tau, \varepsilon))$ and $u(\tau, \varepsilon) = u(\tau + 2\pi, \varepsilon)$ and $\omega(\varepsilon)$ is the unknown frequency. Periodic solutions $u(\tau, \varepsilon)$ of equation (8) are searched in the form of a particular Fourier series, which coefficients depend on the parameter ε

$$u_1(\tau, \varepsilon) = \sum_{k=0}^K (p_{1k}(\varepsilon)\cos k\tau + r_{1k}(\varepsilon)\sin k\tau). \quad (9)$$

Moreover, $p_{1k}(\varepsilon)$, $r_{1k}(\varepsilon)$, $\eta(\varepsilon)$ and $\omega(\varepsilon)$ are developed into the power series of the parameter ε

[†]In the notation used in this chapter this character denotes the differentiation.

$$\begin{aligned}
 p_{ik}(\varepsilon) &= p_{ik0} + p'_{ik}\varepsilon + \frac{1}{2} p''_{ik}\varepsilon^2 + \dots, \\
 r_{ik}(\varepsilon) &= r_{ik0} + r'_{ik}\varepsilon + \frac{1}{2} r''_{ik}\varepsilon^2 + \dots, \\
 \eta(\varepsilon) &= \eta_c + \eta'\varepsilon + \frac{1}{2} \eta''\varepsilon^2 + \dots, \\
 \omega(\varepsilon) &= \omega_c + \omega'\varepsilon + \frac{1}{2} \omega''\varepsilon^2 + \dots,
 \end{aligned} \tag{10}$$

where in the critical point $p_{ik0} = r_{ik0} = 0$. The solution $u_i(\tau, \varepsilon)$ is searched in the form

$$u_i(\tau, \varepsilon) = u_i'(\varepsilon) + \frac{1}{2} u_i''(\tau) \varepsilon^2 + \dots, \tag{11}$$

where

$$u_i^{(*)}(\tau) = \sum_{k=1}^K (p_{ik}^{(*)} \cos k\tau + r_{ik}^{(*)} \sin k\tau). \tag{12}$$

Because the system (8) is autonomous we set a $r_{11}(\varepsilon) = 0$. Now one can act in one of two ways. Either introduce equations (9) to (12) into (8) and then by comparing the terms with the same powers of ε obtain a linear differential equations set, or obtain these equations by successive differentiation of (8). Employing the latter possibility one obtains

$$\omega' u_{i\tau} + \omega_c u'_{i\tau} = \frac{\partial K'_1}{\partial u_1} \frac{\partial u_1}{\partial \varepsilon} + \frac{\partial K'_1}{\partial u_2} \frac{\partial u_2}{\partial \varepsilon} + \frac{\partial K'_1}{\partial \eta} \eta',$$

$$\omega'' u_{i\tau} + 2\omega' u_{i\tau} + \omega_c u''_{i\tau} = \frac{\partial^2 K'_1}{\partial (u_1)^2} \left(\frac{\partial u_1}{\partial \varepsilon} \right)^2 + \frac{\partial^2 K'_1}{\partial (u_2)^2} \left(\frac{\partial u_2}{\partial \varepsilon} \right)^2$$

$$\begin{aligned}
& + 2 \frac{\partial^2 K'_1}{\partial u_1 \partial u_2} \frac{\partial u_1}{\partial \epsilon} \frac{\partial u_2}{\partial \epsilon} + 2 \frac{\partial^2 K'_1}{\partial u_1 \partial \eta} \frac{\partial u_1}{\partial \epsilon} \eta' + 2 \frac{\partial^2 K'_1}{\partial u_2 \partial \eta} \frac{\partial u_2}{\partial \epsilon} \eta' \\
& + \frac{\partial^2 K'_1}{\partial (u_2)^2} \left(\frac{\partial u_2}{\partial \epsilon} \right)^2 + \frac{\partial K'_1}{\partial \eta} \eta'' + \frac{\partial K'_1}{\partial u_1} \frac{\partial^2 u_1}{\partial \epsilon^2} + \frac{\partial K'_1}{\partial u_2} \frac{\partial^2 u_2}{\partial \epsilon^2} \\
& + \frac{\partial^2 K'_1}{\partial \eta^2} (\eta')^2,
\end{aligned}$$

$$\omega''' u_{1\tau} + 3\omega'' u_{1\tau} + 3\omega' u_{1\tau} + \omega_c u_{1\tau} = \frac{\partial K'_1}{\partial u_1} \frac{\partial^3 u_1}{\partial \epsilon^3} + \frac{\partial K'_1}{\partial u_2}$$

$$\begin{aligned}
& \cdot \frac{\partial^3 u_2}{\partial \epsilon^3} + 3 \frac{\partial^2 K'_1}{\partial u_1 \partial \eta} \frac{\partial u_1}{\partial \epsilon} \eta'' + 3 \frac{\partial^2 K'_1}{\partial u_2 \partial \eta} \frac{\partial u_2}{\partial \epsilon} \eta'' + 3 \frac{\partial^2 K'_1}{\partial u_1 \partial \eta} \frac{\partial^2 u_1}{\partial \epsilon^2} \eta' \\
& + 3 \frac{\partial^2 K'_1}{\partial u_2 \partial \eta} \frac{\partial^2 u_2}{\partial \epsilon^2} \eta' + 6 \frac{\partial^3 K'_1}{\partial u_1 \partial u_2 \partial \eta} \frac{\partial u_1}{\partial \epsilon} \frac{\partial u_2}{\partial \epsilon} \eta' + 3 \frac{\partial^3 K'_1}{\partial u_1 \partial \eta^2} \\
& \cdot \frac{\partial u_1}{\partial \epsilon} (\eta')^2 + 3 \frac{\partial^3 K'_1}{\partial u_2 \partial \eta^2} \frac{\partial u_2}{\partial \epsilon} (\eta')^2 + 3 \frac{\partial^2 K'_1}{\partial u_1 \partial u_2} \frac{\partial^2 u_1}{\partial \epsilon^2} \frac{\partial u_2}{\partial \epsilon} \\
& + 3 \frac{\partial^2 K'_1}{\partial u_1 \partial u_2} \frac{\partial^2 u_2}{\partial \epsilon^2} \frac{\partial u_1}{\partial \epsilon} + 3 \frac{\partial^2 K'_1}{\partial (u_1)^2} \frac{\partial^2 u_1}{\partial \epsilon^2} \frac{\partial u_1}{\partial \epsilon} + 3 \frac{\partial^2 K'_1}{\partial (u_2)^2} \\
& \cdot \frac{\partial^2 u_2}{\partial \epsilon^2} \frac{\partial u_2}{\partial \epsilon} + \frac{\partial^3 K'_1}{\partial (u_1)^3} \left(\frac{\partial u_1}{\partial \epsilon} \right)^3 + \frac{\partial^3 K'_1}{\partial (u_2)^3} \left(\frac{\partial u_2}{\partial \epsilon} \right)^3 + 3 \frac{\partial^3 K'_1}{\partial (u_1)^2 \partial u_2}
\end{aligned}$$

$$\begin{aligned} & \cdot \left(\frac{\partial u_1}{\partial \varepsilon} \right)^2 \frac{\partial u_2}{\partial \varepsilon} + 3 \frac{\frac{\partial^3 K_1}{\partial u_1 \partial (u_2)^2}}{\partial u_1 \partial (u_2)^2} \left(\frac{\partial u_2}{\partial \varepsilon} \right)^2 \frac{\partial u_1}{\partial \varepsilon}, \\ & \vdots \qquad \qquad \qquad \vdots \end{aligned} \quad (13)$$

Solving equations (13) successively, we obtain (9) and (10). Solutions $u \in \mathbb{R}^2$ obtained in this way are denoted as u_b and put in the second subsystem of equations (4)

$$\omega_c v_\tau = S_u(\eta) u_b + S_v(\eta) v + \tilde{S}(\eta, u_b, v), \quad (14)$$

where ω , η and u_b are known. In order to obtain v , this system of equations can be solved by a perturbation method.

1.3. Biparameter Hopf Bifurcation

Consider the case when $\eta = (\mu, \delta)$. Taking $\tau = \omega t$ into account one obtains from (1)

$$\omega x_\tau = F(x, \eta). \quad (15)$$

Periodical bifurcation solutions are searched in the form

$$\begin{aligned} x_i(\tau; \mu, \varepsilon) = & x_i^\varepsilon(\tau) + \frac{1}{2} x_i^\varepsilon{}^2(\tau) \varepsilon^2 + \frac{1}{6} x_i^\varepsilon{}^3(\tau) \varepsilon^3 + x_i^{\mu \varepsilon}(\tau) \\ & \cdot \mu \varepsilon + \frac{1}{2} x_i^{\mu^2 \varepsilon}(\tau) \varepsilon \mu^2 + \frac{1}{2} x_i^{\varepsilon^2 \mu}(\tau) \varepsilon^2 \mu + \dots \end{aligned} \quad (16)$$

where ε is a certain perturbation parameter connected with the amplitude of oscillations. Every component of the series (16) is searched in the form

$$x_i^{(*)}(\tau) = \sum_{m=0}^M (p_{im}^{(*)} \cos m\tau + r_{im}^{(*)} \sin m\tau). \quad (17)$$

Coefficients p_{im} , r_{im} , frequency ω and δ are searched in the form of the following series

$$\begin{aligned} p_{im}(\varepsilon, \mu) &= p_{imc} + p_{im}^{\varepsilon} \varepsilon + \frac{1}{2} p_{im}^{\varepsilon^2} \varepsilon^2 + \frac{1}{6} p_{im}^{\varepsilon^3} \varepsilon^3 + p_{im}^{\mu \varepsilon} \mu \varepsilon \\ &+ \frac{1}{2} p_{im}^{\mu^2 \varepsilon} \mu^2 \varepsilon + \frac{1}{2} p_{im}^{\mu \varepsilon^2} \mu \varepsilon^2 + \dots, \\ r_{im}(\varepsilon, \mu) &= r_{imc} + r_{im}^{\varepsilon} \varepsilon + \frac{1}{2} r_{im}^{\varepsilon^2} \varepsilon^2 + \frac{1}{6} r_{im}^{\varepsilon^3} \varepsilon^3 + r_{im}^{\mu \varepsilon} \mu \varepsilon \\ &+ \frac{1}{2} r_{im}^{\mu^2 \varepsilon} \mu^2 \varepsilon + \frac{1}{2} r_{im}^{\mu \varepsilon^2} \mu \varepsilon^2 + \dots, \\ \omega(\mu, \varepsilon) &= \omega_c + \omega_{\varepsilon} \varepsilon + \frac{1}{2} \omega_{\varepsilon \varepsilon} \varepsilon^2 + \frac{1}{6} \omega_{\varepsilon \varepsilon \varepsilon} \varepsilon^3 + \omega_{\mu \varepsilon} \mu \varepsilon \\ &+ \frac{1}{2} \omega_{\mu \varepsilon^2} \mu \varepsilon^2 + \frac{1}{2} \omega_{\varepsilon \mu^2} \varepsilon \mu^2 + \omega_{\mu \mu} \mu + \frac{1}{2} \omega_{\mu \mu \mu} \mu^2 + \dots, \\ \delta(\mu, \varepsilon) &= \delta_c + \delta_{\varepsilon} \varepsilon + \frac{1}{2} \delta_{\varepsilon \varepsilon} \varepsilon^2 + \frac{1}{2} \delta_{\varepsilon \varepsilon \varepsilon} \varepsilon^3 + \delta_{\mu \varepsilon} \mu \varepsilon \\ &+ \frac{1}{2} \delta_{\mu \varepsilon^2} \mu \varepsilon^2 + \frac{1}{2} \delta_{\varepsilon \mu^2} \varepsilon \mu^2 + \delta_{\mu} + \frac{1}{2} \delta_{\mu \mu \mu} \mu^2 + \dots \end{aligned} \quad (18)$$

The subsequent differentiation of (15) in relation to ε and μ gives

$$\omega_{\varepsilon} x_{i\tau} + \omega_c x_{i\tau}^{\varepsilon} = F_{ij} x_j^{\varepsilon},$$

$$\omega_{\varepsilon\varepsilon} x_{i\tau} + 2\omega_{\varepsilon} x_{i\tau}^{\varepsilon} + \omega_c x_{i\tau}^{\varepsilon^2} = F_{ijk} x_j^{\varepsilon} x_k^{\varepsilon} + 2F_{ij\delta} x_j^{\varepsilon} \delta_{\varepsilon} + F_{ij} x_j^{\varepsilon^2},$$

$$\omega_{\varepsilon\mu} x_{i\tau} + \omega_{\mu} x_{i\tau}^{\varepsilon} + \omega_c x_{i\tau}^{\varepsilon\mu} = F_{ij} x_j^{\varepsilon\mu}$$

$$+ F_{ij\delta} \delta_{\mu} x_j^{\varepsilon} + F_{ij\mu} x_j^{\varepsilon} + F_{i\delta\delta} \delta_{\varepsilon} \delta_{\mu} + F_{i\delta\mu} \delta_{\varepsilon},$$

$$\omega_{\varepsilon\varepsilon\varepsilon} x_{i\tau} + 3\omega_{\varepsilon\varepsilon} x_{i\tau}^{\varepsilon} + 3\omega_{\varepsilon} x_{i\tau}^{\varepsilon^2} + \omega_c x_{i\tau}^{\varepsilon^3} = F_{ijkl} x_j^{\varepsilon} x_k^{\varepsilon} x_l^{\varepsilon}$$

$$+ 3F_{ijk\delta} x_j^{\varepsilon} x_k^{\varepsilon} \delta_{\varepsilon} + 3F_{ijk} x_j^{\varepsilon} x_k^{\varepsilon^2} + 3F_{ij\delta\delta} x_j^{\varepsilon} (\delta_{\varepsilon})^2$$

$$+ 3F_{ij\delta} x_j^{\varepsilon^2} \delta_{\varepsilon} + 3F_{ij\delta} x_j^{\varepsilon} \delta_{\varepsilon\varepsilon} + F_{ij} x_j^{\varepsilon^3}, \quad (19)$$

$$\omega_{\varepsilon\mu^2} x_{i\tau} + \omega_{\mu\mu} x_{i\tau}^{\varepsilon} + 2\omega_{\mu} x_{i\tau}^{\varepsilon} + \omega_c x_{i\tau}^{\varepsilon\mu^2} = F_{ijk} x_j^{\varepsilon} x_k^{\mu^2}$$

$$+ 2F_{ij\mu} x_j^{\varepsilon\mu} + 2F_{ij\delta} \delta_{\mu} x_j^{\varepsilon\mu} + F_{ij} x_j^{\varepsilon\mu^2} + 2F_{ij\delta\mu} \delta_{\mu} x_j^{\varepsilon}$$

$$+ F_{ij\delta\delta} (\delta_{\mu})^2 x_j^{\varepsilon} + F_{ij\delta} \delta_{\mu\mu} x_j^{\varepsilon} + F_{ij\mu\mu} x_j^{\varepsilon} + F_{ij\delta} \delta_{\varepsilon} x_j^{\mu^2}$$

$$+ F_{i\delta\delta\delta} \delta_{\varepsilon} (\delta_{\mu})^2 + 2F_{i\delta\delta\mu} \delta_{\varepsilon} \delta_{\mu} + F_{i\delta\delta} \delta_{\varepsilon\mu} \delta_{\mu} + F_{i\delta\delta} \delta_{\varepsilon} \delta_{\mu\mu}$$

$$+ F_{i\delta\mu\mu} \delta_{\varepsilon} + F_{i\delta\mu} \delta_{\varepsilon} \mu,$$

$$\begin{aligned}
& \omega_{\varepsilon}^2 x_{1\tau}^{\varepsilon} + 2\omega_{\varepsilon\mu} x_{1\tau}^{\varepsilon} + 2\omega_{\varepsilon} x_{1\tau}^{\varepsilon\mu} + \omega_{\mu} x_{1\tau}^{\varepsilon^2} + \omega_{\varepsilon} x_1^{\varepsilon^2\mu} = \\
& F_{ijk\mu} x_j^{\varepsilon} x_k^{\varepsilon} + F_{ijk} x_j^{\varepsilon\mu} x_k^{\varepsilon} + F_{ijk} x_j^{\varepsilon} x_k^{\varepsilon\mu} + F_{ijk\delta\mu} x_j^{\varepsilon} x_k^{\varepsilon} \\
& + 2F_{ij\delta\mu} x_j^{\varepsilon} \delta_{\varepsilon} + 2F_{ij\delta} x_j^{\varepsilon\mu} \delta_{\varepsilon} + 2F_{ij\delta} x_j^{\varepsilon} \delta_{\mu\varepsilon} + 2F_{ij\delta\delta} (\delta_{\mu})^2 x_j^{\varepsilon} \delta_{\varepsilon} \\
& + F_{ij\mu} x_j^{\varepsilon^2} + F_{ij} x_j^{\varepsilon^2\mu} + F_{ij\delta} \delta_{\mu} x_j^{\varepsilon^2},
\end{aligned}$$

where

$$\omega_{\varepsilon\varepsilon\varepsilon} = \frac{\partial^3 \omega}{\partial \varepsilon^3}; \quad \omega_{\varepsilon^2\mu} = \frac{\partial^3 \omega}{\partial \varepsilon^2 \partial \mu}, \quad \dots, \quad F_{ijk} = \frac{\partial^2 F_1}{\partial x_j \partial x_k},$$

$$F_{ijk\delta} = \frac{\partial^3 F_1}{\partial x_j \partial x_k \partial \delta}, \quad \dots$$

Obtained differential perturbation equations (19) are solved by means of the harmonic balance method, having taken into account relations (18) and assumed $r_{11} = 0$.

1.4. Bifurcation into Quasiperiodic Torus

Consider now equations (1) where $\eta \in \mathbb{R}^2$ and where the Jacobian matrix has a block diagonal form

$$DF_{\mathbf{x}}(\eta_{\mathbf{c}}, \mathbf{x}_0) = \text{block diag} \{ C_1, C_2, P_1, P_2, \dots, R_1, R_2, \dots \}$$

with real elements. The blocks are given

$$G_1 = \begin{bmatrix} 0 & \omega_{1c} \\ -\omega_{1c} & 0 \end{bmatrix},$$

$$G_2 = \begin{bmatrix} 0 & \omega_{2c} \\ -\omega_{2c} & 0 \end{bmatrix},$$

(20)

$$P_i = \begin{bmatrix} \alpha_i & \omega_i \\ -\omega_i & \alpha_i \end{bmatrix}, \quad i = 1, 2, \dots,$$

$$R_j = [\alpha_j], \quad j = 1, 2, \dots,$$

while $\alpha_i < 0$ and $\alpha_j < 0$ for $\eta < \eta_c$.

Let $\lambda_{1,2}(\eta_1) = \alpha(\eta_1) \pm i \omega_1(\eta_1)$ and $\lambda_{3,4} = \beta(\eta_2) \pm i \omega_2(\eta_2)$ be the two pairs of eigenvalues of DF_x that cross the imaginary axis at $\eta_1 = \eta_{1c}$ and $\eta_2 = \eta_{2c}$ with non-zero velocity as η_1 and η_2 are increased. Then for $\eta_1 = \eta_{1c}$ and $\eta_2 = \eta_{2c}$ we have

$$\lambda_{1,2}(\eta_{1c}) = \pm i \omega_{1c},$$

(21)

$$\lambda_{3,4}(\eta_{2c}) = \pm i \omega_{2c}$$

and the real parts of $\lambda_{1,2}$ and $\lambda_{3,4}$ should satisfy

$$\left. \frac{d\alpha(\eta_1)}{d\eta_1} \right|_{\eta_1 = \eta_{1c}} \neq 0, \quad (22)$$

$$\left. \frac{d\beta(\eta_2)}{d\eta_2} \right|_{\eta_2 = \eta_{2c}} \neq 0.$$

We consider the non-resonant case, i. e. when $\frac{\omega_{1c}}{\omega_{2c}} \neq \frac{k}{l}$, where k, l are integer numbers.

Because of the analogy to the Hopf bifurcation this case will be referred to as called double Hopf bifurcation.

The bifurcated solutions of (1) are assumed to be in the parametric form given by

$$x_i = x_i(t, \varepsilon_1, \varepsilon_2). \quad (23)$$

The solutions (23) are represented by a double Fourier series

$$\begin{aligned} x_i = & \sum_{k=0}^K p_{iko}^s(\varepsilon_1, \varepsilon_2) \sin k\tau_1 + \sum_{k=0}^K p_{iko}^c(\varepsilon_1, \varepsilon_2) \cos k\tau_1 \\ & + \sum_{l=0}^L p_{iol}^c(\varepsilon_1, \varepsilon_2) \cos l\tau_2 + \sum_{l=0}^L p_{iol}^s(\varepsilon_1, \varepsilon_2) \sin l\tau_2 \\ & + \frac{1}{2} \sum_{k=1}^K \sum_{l=1}^L p_{ikl}^{cc}(\varepsilon_1, \varepsilon_2) [\cos(k\tau_1 + l\tau_2) + \cos(k\tau_1 \\ & - l\tau_2)] + \frac{1}{2} \sum_{k=1}^K \sum_{l=1}^L p_{ikl}^{cs}(\varepsilon_1, \varepsilon_2) [\cos(k\tau_1 + l\tau_2) - \cos(k\tau_1 \\ & - l\tau_2)] + \frac{1}{2} \sum_{k=1}^K \sum_{l=1}^L p_{ikl}^{sc}(\varepsilon_1, \varepsilon_2) [\sin(k\tau_1 + l\tau_2) + \sin(k\tau_1 \\ & - l\tau_2)] + \frac{1}{2} \sum_{k=1}^K \sum_{l=1}^L p_{ikl}^{ss}(\varepsilon_1, \varepsilon_2) [\sin(k\tau_1 + l\tau_2) - \sin(k\tau_1 \\ & - l\tau_2)] \end{aligned}$$

$$\begin{aligned}
& - 1\tau_2)] + p_{ikl}^{cs}(\varepsilon_1, \varepsilon_2) [\sin(k\tau_1 + 1\tau_2) + \sin(1\tau_2 - k\tau_1)] \\
& + p_{ikl}^{sc}(\varepsilon_1, \varepsilon_2) [\sin(k\tau_1 + 1\tau_2) + \sin(k\tau_1 - 1\tau_2)] \\
& + p_{ikl}^{ss}(\varepsilon_1, \varepsilon_2) [\cos(k\tau_1 - 1\tau_2) - \cos(k\tau_1 + 1\tau_2)] .
\end{aligned} \tag{24}$$

The coefficients $p_{i(\cdot)}^{(*)}(\varepsilon_1, \varepsilon_2)$, $r_{i(\cdot)}^{(*)}(\varepsilon_1, \varepsilon_2)$, the frequencies $\omega_1(\varepsilon_1, \varepsilon_2)$, $\omega_2(\varepsilon_1, \varepsilon_2)$ and the bifurcation parameters $\eta_1(\varepsilon_1, \varepsilon_2)$, $\eta_2(\varepsilon_1, \varepsilon_2)$ are searched in the form

$$\begin{aligned}
p_{i(\cdot)}^{(*)}(\varepsilon_1, \varepsilon_2) &= p_{ic(\cdot)}^{(*)} + \varepsilon_1 p_{i(\cdot)}^{(*)'} + \frac{1}{2} \varepsilon_1^2 p_{i(\cdot)}^{(*)''} + \frac{1}{6} \varepsilon_1^3 p_{i(\cdot)}^{(*)'''} \\
&+ \varepsilon_2 p_{i(\cdot)}^{(*)'} + \frac{1}{2} \varepsilon_2^2 p_{i(\cdot)}^{(*)''} + \frac{1}{6} \varepsilon_2^3 p_{i(\cdot)}^{(*)'''} + \varepsilon_1 \varepsilon_2 p_{i(\cdot)}^{(*)'} + \frac{1}{2} \varepsilon_1^2 \varepsilon_2 p_{i(\cdot)}^{(*)''} \\
&+ \frac{1}{2} \varepsilon_2^2 \varepsilon_1 p_{i(\cdot)}^{(*)''} + \dots ,
\end{aligned} \tag{25}$$

$$\begin{aligned}
r_{i(\cdot)}^{(*)}(\varepsilon_1, \varepsilon_2) &= r_{ic(\cdot)}^{(*)} + \varepsilon_1 r_{i(\cdot)}^{(*)'} + \frac{1}{2} \varepsilon_1^2 r_{i(\cdot)}^{(*)''} + \frac{1}{6} \varepsilon_1^3 r_{i(\cdot)}^{(*)'''} \\
&+ \varepsilon_2 r_{i(\cdot)}^{(*)'} + \frac{1}{2} \varepsilon_2^2 r_{i(\cdot)}^{(*)''} + \frac{1}{6} \varepsilon_2^3 r_{i(\cdot)}^{(*)'''} + \varepsilon_1 \varepsilon_2 r_{i(\cdot)}^{(*)'} + \frac{1}{2} \varepsilon_1^2 \varepsilon_2 r_{i(\cdot)}^{(*)''} \\
&+ \frac{1}{2} \varepsilon_2^2 \varepsilon_1 r_{i(\cdot)}^{(*)''} + \dots ,
\end{aligned}$$

$$\omega_1 = \frac{d\tau_1}{dt} = \omega_{1c} + \varepsilon_1 \omega_1' + \frac{1}{2} \varepsilon_1^2 \omega_1'' + \frac{1}{6} \varepsilon_1^3 \omega_1''' + \varepsilon_2 \omega_1'$$

$$+ \frac{1}{2} \varepsilon_2^2 \omega_1'' + \frac{1}{6} \varepsilon_2^3 \omega_1''' + \varepsilon_1 \varepsilon_2 \omega_1' + \frac{1}{2} \varepsilon_1^2 \varepsilon_2 \omega_1'' + \frac{1}{2} \varepsilon_1 \varepsilon_2^2 \omega_1'' + \dots,$$

$$\omega_2 = \frac{d\tau_2}{dt} = \omega_{2c} + \varepsilon_1 \omega_2' + \frac{1}{2} \varepsilon_1^2 \omega_2'' + \frac{1}{6} \varepsilon_1^3 \omega_2''' + \varepsilon_2 \omega_2'$$

$$+ \frac{1}{2} \varepsilon_2^2 \omega_2'' + \frac{1}{6} \varepsilon_2^3 \omega_2''' + \varepsilon_1 \varepsilon_2 \omega_2' + \frac{1}{2} \varepsilon_1^2 \varepsilon_2 \omega_2'' + \frac{1}{2} \varepsilon_1 \varepsilon_2^2 \omega_2'' + \dots,$$

$$\eta_1 = \eta_{1c} + \varepsilon_1 \eta_1' + \frac{1}{2} \varepsilon_1^2 \eta_1'' + \frac{1}{6} \varepsilon_1^3 \eta_1''' + \varepsilon_2 \eta_1' + \frac{1}{2} \varepsilon_2^2 \eta_1''$$

$$+ \frac{1}{6} \varepsilon_2^3 \eta_1''' + \varepsilon_1 \varepsilon_2 \eta_1' + \frac{1}{2} \varepsilon_1^2 \varepsilon_2 \eta_1'' + \frac{1}{2} \varepsilon_1 \varepsilon_2^2 \eta_1'' + \dots,$$

$$\eta_2 = \eta_{2c} + \varepsilon_1 \eta_2' + \frac{1}{2} \varepsilon_1^2 \eta_2'' + \frac{1}{6} \varepsilon_1^3 \eta_2''' + \varepsilon_2 \eta_2' + \frac{1}{2} \varepsilon_2^2 \eta_2''$$

$$+ \frac{1}{6} \varepsilon_2^3 \eta_2''' + \varepsilon_1 \varepsilon_2 \eta_2' + \frac{1}{2} \varepsilon_1^2 \varepsilon_2 \eta_2'' + \frac{1}{2} \varepsilon_1 \varepsilon_2^2 \eta_2'' + \dots.$$

By introducing (24) and (25) into (1) and equating the expressions of the same powers $\varepsilon_1^k \varepsilon_2^l$ ($k, l = 0, 1, 2, \dots$) a sequence of linear differential equations are obtained, which are solvable by means of the harmonic balance method. Because the system is autonomous it is possible to obtain the dependencies of unknown frequencies and bifurcation parameters on amplitudes of oscillations in this way. Independent small perturbation parameters are connected with amplitudes of oscillations.

1.5. Hopf Bifurcation in Duffing's Oscillator

1.5.1. A vibrating system

Consider the vibrating mechanical system with one degree of freedom

freedom

$$m \frac{d^2 x}{dt^2} - (c_1 - c_2 x^2) \frac{dx}{dt} + kx + k_1 x^3 = P_0 \cos \omega_1 t, \quad (26)$$

where m is the mass of the vibrating body, c_1 and c_2 are damping factors, k and k_1 - rigidity factors, and P_0 and ω_1 are respectively, amplitude and external excitation frequency.

The eq. (26) assumes the dimensionless form

$$\frac{d^2 y}{d\tau^2} - \eta \frac{dy}{d\tau} + \delta y^2 \frac{dy}{d\tau} + \omega_c^2 y + \xi y^3 = \lambda \cos \tau, \quad (27)$$

where

$$\tau = \omega_1 t, \quad x = \frac{P_{\max}}{m \omega_1^2} y, \quad \eta = \frac{c_1}{m \omega_1}, \quad \delta = \frac{c_2 P_{\max}^2}{m^3 \omega_1^5}, \quad \alpha_0^2 = \frac{k}{m},$$

$$\omega_c^2 = \frac{\alpha_0^2}{\omega_1^2}, \quad \xi = \frac{k_1 P_{\max}^2}{m^3 \omega_1^6}, \quad \lambda = \frac{P_0}{P_{\max}}. \quad (28)$$

The eq. (27) is a particular type of the Van der Pol-Duffing equation, and for $\eta = 0$ it circumscribes the vibrations of Duffing's oscillator. It can be presented in the form

$$\frac{dx_1}{d\tau} = x_2 - \frac{\delta x_1^3}{3} + \eta x_1, \quad (29)$$

$$\frac{dx_2}{d\tau} = -\omega_c^2 x_1 - \xi x_1^3 + \lambda \cos \tau,$$

where $x_1 = y$. The characteristic equation obtained from (29), in the case of lack of external excitation, has the roots

$$\epsilon_{1,2} = 1/2(\eta \pm (\eta^2 - 4\omega_c^2)^{1/2}).$$

For $\eta_c = 0$, at the critical point

$\bar{\epsilon}_{1,2} = \pm i\omega_c$ and $d\bar{\epsilon}_{1,2}/d\eta|_c = 1/2$ conditions of occurrence of the Hopf bifurcation are fulfilled.

1.5.2. Non-resonance case

Let $\lambda = \epsilon\lambda_1$, then the eq. (29) will have the form

$$\begin{aligned}\frac{dx_1^{(1)}}{d\tau} &= x_2^{(1)} - \frac{\delta(x_1^{(1)})^3}{3} + \eta x_1^{(1)}, \\ \frac{dx_2^{(1)}}{d\tau} &= -\omega_c^2 x_1^{(1)} - \xi(x_1^{(1)})^3 + \epsilon\lambda_1 \cos\tau.\end{aligned}\quad (30)$$

The solutions of (30) are searched in the form of series

$$x_i^{(1)} = x_{ic}^{(1)} + \epsilon x_i^{(1)'} + \frac{1}{2} \epsilon^2 x_i^{(1)''} + \frac{1}{6} \epsilon^3 x_i^{(1)'''} + \dots, \quad (31)$$

$$i = 1, 2.$$

$$\eta = \eta_c + \epsilon\eta' + \frac{1}{2} \epsilon^2 \eta'' + \frac{1}{6} \epsilon^3 \eta''' + \dots, \quad (32)$$

$$\omega = \omega_c + \epsilon\omega' + \frac{1}{2} \epsilon^2 \omega'' + \frac{1}{6} \epsilon^3 \omega''' + \dots. \quad (33)$$

In the critical point $\eta_c = x_{ic}^{(1)} = 0$. The bifurcation solutions $x_i^{(1)}(\tau, \epsilon)$ are represented by the Fourier series

$$\begin{aligned}
x_i^{(1)} = & \sum_{k=0}^K (p_{ik0}^s(\epsilon) \sin k\tau_1 + p_{ik0}^c \cos k\tau_1) \\
& + \sum_{l=0}^L (p_{iol}^c(\epsilon) \cos l\tau + p_{iol}^s(\epsilon) \sin l\tau) + \frac{1}{2} \sum_{l=1}^L \sum_{k=1}^K \{ p_{ikl}^{cc}(\epsilon) \\
& [\cos(k\tau_1 + l\tau) + \cos(k\tau_1 - l\tau)] + p_{ikl}^{cs}(\epsilon) [\sin(k\tau_1 \\
& + l\tau) + \sin(l\tau - k\tau_1)] + p_{ikl}^{sc}(\epsilon) [\sin(k\tau_1 + l\tau) \\
& + \sin(k\tau_1 - l\tau)] + p_{ikl}^{ss}(\epsilon) [\cos(k\tau_1 - l\tau) - \cos(k\tau_1 \\
& + l\tau)] \} , \tag{34}
\end{aligned}$$

where: $\tau_1 = \omega\tau$,

$$p_i^{(*)} = p_{ic}^{(*)} + \epsilon p_{i1}^{(*)'} + \frac{1}{2} \epsilon^2 p_{i1}^{(*)''} + \frac{1}{6} \epsilon^3 p_{i1}^{(*)'''} + \dots \tag{35}$$

The following parameter-frequency relations and the dependence of the bifurcation parameter on amplitude are obtained

$$\eta = \frac{\delta}{2} \left(\frac{1}{2} A^2 + \frac{\lambda^2}{(\omega_c^2 - 1)^2} \right) , \tag{36}$$

$$\omega = \omega_c + \frac{3}{4} \frac{\epsilon}{\omega_c} \left(\frac{1}{2} A^2 + \frac{\lambda^2}{(\omega_c^2 - 1)^2} \right) , \tag{37}$$

$$\text{where: } A^2 = (\epsilon p_{110}^{c'})^2 + (\epsilon p_{110}^{s'})^2.$$

1.5.3. Main resonance

Let $\lambda = \xi^3 \lambda_2$, then equation (29) gives

$$\frac{dx_1^{(2)}}{d\tau} = x_2^{(2)} - \frac{\xi(x_1^{(2)})^3}{3} + \eta x_1^{(2)}, \quad (38)$$

$$\frac{dx_2^{(2)}}{d\tau} = -\omega_c^2 x_1^{(2)} - \xi(x_1^{(2)})^3 + \xi^3 \lambda_2 \cos \tau.$$

Let us assume that frequency ω_c and the external excitation frequency differ from each other only slightly, i. e. let $\omega_c \cong 1 - 1/2a'$, where $a' = \xi^2 a$. Now we act analogously to the non-resonance case

$$\eta = \frac{\xi}{4} A^2 - \frac{\lambda}{A} \sin \varphi, \quad (39)$$

$$\omega = \omega_c - \frac{1}{2} \frac{\lambda \cos \varphi}{A} + \frac{3}{8} \xi A^2, \quad (40)$$

where: $p_{110}^{s'} \xi = A \sin \varphi$; $p_{110}^{c'} \xi = A \cos \varphi$.

1.5.4 Resonance of the n-th order

In this case it is assumed that the following relation exists between the frequency and external excitation frequency: $\omega_c \cong 1/n(1 - 1/2a')$. The equation (29), after taking

$\lambda = \varepsilon \lambda_1$, $\tau = n\tau_2$, gives

$$\frac{dx_1^{(3)}}{d\tau} = x_2^{(3)} - n\frac{\delta}{3}(x_1^{(3)})^3 + n\eta x_1^{(3)}, \quad (41)$$

$$\frac{dx_2^{(3)}}{d\tau} = -x_1^{(3)} + ax_1^{(3)} - n^2\xi(x_1^{(3)})^3 + \varepsilon n^2\lambda_1 \cos n\tau_2.$$

In this case we obtain

$$\eta = \frac{1}{2} \delta \left(\frac{1}{2} A^2 + \frac{\lambda^2 n^4}{(1-n^2)^2} \right), \quad (42)$$

$$\omega = n\omega_c + \frac{1}{2} n^2 \xi \left(\frac{3}{4} A^2 + \frac{3}{2} \frac{\lambda^2 n^4}{(1-n^2)^2} \right). \quad (43)$$

1.5.5. Concluding remarks

On the example of the forced Van der Pol-Duffing's oscillator, analytical method of determining the post-critical family of solutions after Hopf bifurcation in nonlinear nonautonomous oscillators with one bifurcation parameter is presented. It is assumed that Hopf's conditions are fulfilled for a certain value of the bifurcation parameter when no forcing exists in the system. The bifurcated solutions are sought in the form of a particular Fourier series. The practical application of the series consists in solving perturbation equations of the first, second and k-th order by means of the harmonic balancing method. The exemplary calculations have been reduced to $k=3$, as an approximate bifurcation solution

is determined in principle near the critical point. In the investigated case the amplitude is connected with the small perturbation parameter. The exciting force amplitudes are of the order of the amplitude A , or are smaller.

In the case of no resonance as well as in that of the resonance of n -th order, the parameter ξ has no influence on the dependence of bifurcation parameter, amplitude, amplitude of the exciting force, whereas the parameter δ has no influence on the dependence of amplitude, frequency, amplitude of the exciting force.

1.6. Hopf Bifurcation in Nonstationary Nonlinear Systems

1.6.1. Example without external force

Consider a vibrating mechanical system with one degree of freedom — a rotor with unequal moments of inertia of its cross-section, with its mass concentrated in its center. The equation of motion of the system has the form

$$m\ddot{y} + \frac{1}{2}(k_1 + k_2 + (k_1 - k_2)\cos 2\omega t)y + k_0 y^3 = 0, \quad (44)$$

where m is the concentrated mass, k_0 is the non-linear rigidity, k_1, k_2 are the rotor rigidities, and ω is the rotation frequency of the rotor. After a change of variables, a dimensionless form of the equation is obtained,

$$d^2x/d\tau^2 + (\xi^2 + \mu\cos 2\tau)x + \xi x^3 = 0, \quad (45)$$

where

$$x = y(k_1/k_0)^{-1/2}, \quad \tau = \omega t, \quad \xi = k_1/(m\omega^2),$$

$$\delta^2 = \frac{1}{2}(k_1 + k_2)/m\omega^2, \quad \mu = \frac{1}{2}(k_1 - k_2)/(k_1 + k_2)\omega^2.$$

Equation (45) can be expressed as a system of two differential equations of the first order,

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\delta^2 x_1 - \mu x_1 \cos 2\tau - \varepsilon x_1^3, \quad (46)$$

where $x_1 = y$.

The bifurcation solutions are sought in the form of a series in the perturbation parameter ε connected with the vibration amplitude, of the form

$$x_1 = x_1^0 + \varepsilon x_1' + \frac{1}{2} \varepsilon^2 x_1'' + \frac{1}{6} \varepsilon^3 x_1''' + \dots, \quad (47)$$

and the frequency δ^2 and the parameter μ also can be developed into power series in the perturbation parameter,

$$\delta^2 = \delta_c^2 + \varepsilon \delta' + \frac{1}{2} \varepsilon^2 \delta'' + \frac{1}{6} \varepsilon^3 \delta''' + \dots, \quad (48)$$

$$\mu = \mu_c + \varepsilon \mu' + \frac{1}{2} \varepsilon^2 \mu'' + \frac{1}{6} \varepsilon^3 \mu''' + \dots, \quad (49)$$

where $\delta_c^2 = n^2$ and $\mu_c = 0$. After substituting equations (47) - (49) into equations (46) and developing $x_1^{(k)}$ into Fourier series, harmonic balancing is performed at ε and $\sin n\tau$, $\cos n\tau$ and the following is obtained:

$$p_{2n0}^{s'} = -np_{1n0}^{c'}, \quad p_{2n0}^{c'} = np_{1n0}^{s'}. \quad (50)$$

Then

$$x_1^3 = \varepsilon^3 ((p_{1n0}^{c'})^3 (\frac{3}{4} \cos n\tau + \frac{1}{4} \cos 3n\tau) + \frac{3}{4} (p_{1n0}^{c'})^2 \cdot p_{1n0}^{s'} (\sin 3n\tau + \sin n\tau) + \frac{3}{4} (p_{1n0}^{s'})^2 p_{1n0}^{c'} (\cos n\tau - \cos 3n\tau))$$

$$+ (p_{1n0}^{s'})^3 \left(\frac{3}{4} \sin n\tau - \frac{1}{4} \sin 3n\tau \right). \quad (51)$$

After harmonic balancing at ϵ^3 , one obtains, for $n \neq 2$,

$$\begin{aligned} n^2 p_{1n0}^{c'} \delta'' &= \frac{3}{4} ((p_{1n0}^{c'})^3 + (p_{1n0}^{s'})^2 p_{1n0}^{c'}), \\ n^2 p_{1n0}^{s'} \delta'' &= \frac{3}{4} ((p_{1n0}^{s'})^3 + (p_{1n0}^{c'})^2 p_{1n0}^{s'}). \end{aligned} \quad (52)$$

The following is the solution of equations (52):

$$\mu'' = p_{1n0}^{c'} = 0, \quad \delta''_{(1)} = \frac{3}{4} (1/n^2) \xi (p_{1n0}^{s'})^2; \quad (53)$$

$$\mu'' = p_{1n0}^{s'} = 0, \quad \delta''_{(2)} = \frac{3}{4} (1/n^2) \xi (p_{1n0}^{c'})^2. \quad (54)$$

On the other hand, for $n = 2$ the following is obtained:

$$p_{120}^{c'} = \delta'' = 0, \quad \mu'' = -\frac{3}{16} \xi (p_{120}^{s'})^2; \quad (55)$$

$$p_{120}^{s'} = \delta'' = 0, \quad \mu'' = -\frac{3}{16} \xi (p_{120}^{c'})^2. \quad (56)$$

1.6.2. Example including external force

Let now a harmonic force have an effect on the system considered above. In this case the equation of motion has the form

$$\begin{aligned} m\ddot{y} + \frac{1}{2}(k_1 + k_2 + (k_1 - k_2)\cos 2\omega t)y + k_0 y^3 &= \\ &= P_0 \cos \omega_1 t. \end{aligned} \quad (57)$$

Let

$$\begin{aligned}\tau_* &= \omega t, \quad x = y(k_1/k_0)^{-1/2}, \quad \lambda_* = (p_0/m\omega^2)(k_0/k_1)^{1/2}, \\ \xi_* &= k_1/(m\omega^2), \quad \delta_*^2 = (k_1 + k_2)/2m\omega^2, \quad \omega_1/\omega = \omega_*, \\ \mu_* &= (k_1 - k_2)/2m\omega^2.\end{aligned}\quad (58)$$

Then equation (57) will have the form

$$d^2x/d\tau^2 + (\delta_*^2 + \mu_* \cos 2\tau_*)x + \xi_* x^3 = \lambda_* \cos \omega_* \tau_*. \quad (59)$$

The system of two differential equations replacing equation (59) is

$$\begin{aligned}\dot{x}_1 &= x_2, \quad \dot{x}_2 = -\delta_*^2 x_1 - \xi_* x_1^3 - \mu_* x_1 \cos 2\tau_* + \lambda_* \cos \omega_* \tau_*,\end{aligned}\quad (60)$$

where $\lambda_* = \epsilon \lambda_{1*}$. The Fourier series in this case is double and has the independent variable $n\tau_*$ and $l\omega_*\tau_*$.

Consider first the non-resonance case ($l\omega_* \neq n$), where $n, l \in \mathbb{N}$. After harmonic balancing at ϵ and $\sin n\tau_*$, $\cos n\tau_*$ and $\sin l\omega_*\tau_*$, $\cos l\omega_*\tau_*$ and after solving the algebraic equations set one has

$$p_{2n0}^{s'} = -np_{1n0}^{c'}, \quad p_{2n0}^{c'} = np_{1n0}^{s'}; \quad (61)$$

$$p_{101}^{s'} = p_{201}^{c'} = 0, \quad p_{101}^{c'} = \lambda_1/(n^2 - l^2\omega_*^2),$$

$$p_{201}^{s'} = -l\lambda_1/(n^2 - l^2\omega_*^2). \quad (62)$$

Then

$$\begin{aligned}
x_1^3 = & (p_{1n0}^{c'})^3 \left(\frac{3}{4} \cos n\tau_* + \frac{1}{4} \cos 3n\tau_* \right) + \frac{3}{4} (p_{1n0}^{c'})^2 \\
& \cdot p_{1n0}^{s'} (\sin 3n\tau_* + \sin n\tau_*) + \frac{3}{4} (p_{1n0}^{s'})^2 p_{1n0}^{c'} \\
& \cdot (\cos n\tau_* - \cos 3n\tau_*) + (p_{1n0}^{s'})^3 \left(\frac{3}{4} \sin n\tau_* \right. \\
& \left. - \frac{1}{4} \sin 3n\tau_* \right) + \frac{3}{2} (p_{1n0}^{c'})^2 p_{101}^{c'} \cos 1\omega_* \tau_* + \frac{3}{4} (p_{1n0}^{c'})^2 \\
& \cdot p_{101}^{c'} (\cos(2n + 1\omega_*)\tau_* + \cos(2n - 1\omega_*)\tau_*) \quad (63) \\
& + \frac{3}{2} p_{1n0}^{c'} p_{1n0}^{s'} p_{101}^{c'} (\sin(2n + 1\omega_*)\tau_* + \sin(2n - 1\omega_*)\tau_*) \\
& + \frac{3}{2} (p_{1n0}^{s'})^2 p_{101}^{c'} \cos 1\omega_* \tau_* - \frac{3}{4} (p_{1n0}^{s'})^2 p_{101}^{c'} (\cos(2n \\
& + 1\omega_*)\tau_* + \cos(2n - 1\omega_*)\tau_*) + \frac{3}{2} p_{1n0}^{c'} (p_{101}^{c'})^2 \cos n\tau_* \\
& + \frac{3}{4} p_{1n0}^{c'} (p_{101}^{c'})^2 (\cos(n + 21\omega_*)\tau_* + \cos(n - 21\omega_*)\tau_*) \\
& + \frac{3}{2} p_{1n0}^{s'} (p_{101}^{c'})^2 \sin n\tau_* + \frac{3}{2} p_{1n0}^{s'} (p_{101}^{c'})^2 (\sin(n + \\
& + 21\omega_*)\tau_* + \sin(n - 21\omega_*)\tau_*) + (p_{101}^{c'})^3 \left(\frac{3}{4} \cos 1\omega_* \tau_* \right. \\
& \left. + \frac{1}{4} \cos 31\omega_* \tau_* \right).
\end{aligned}$$

When comparing the terms at orders ϵ^3 and $\sin n\tau_*$, $\cos n\tau_*$, as well as at $\sin 1\omega_* \tau_*$ and $\cos 1\omega_* \tau_*$ one can obtain μ'' and δ'' and finally, for this case,

$$\mu = 0,$$

$$\xi^2 = \frac{3}{4} \xi \left(-\frac{1}{n^2} \left(\frac{1}{2} (p_{1n0}^{c'} \xi)^2 + \frac{1}{2} (p_{1n0}^{s'} \xi)^2 \right) + (p_{101}^{c'} \xi)^2 \right)$$

$$+ 1^{-2} \omega_*^{-2} ((p_{1n0}^{c'} \epsilon)^2 + (p_{1n0}^{s'} \epsilon)^2 + \frac{1}{2} (p_{101}^{c'} \epsilon)^2)). \quad (64)$$

Let

$$p_{1n0}^{c'} \epsilon = A \cos \varphi, \quad p_{1n0}^{s'} \epsilon = A \sin \varphi. \quad (65)$$

Then the following is obtained from using the expressions in (65):

$$\begin{aligned} \mu &= 0, \\ &\frac{A^2}{2n^2 1^2 \omega_*^2 / (1^2 \omega_*^2 + 2n^2)} + \frac{\lambda_*^2}{2(n^2 - 1^2 \omega_*^2) 1^2 \omega_*^2 n^2 / (2 1^2 \omega_*^2 + n^2)} \\ &- \frac{\delta^2}{4 \xi} = 0. \end{aligned} \quad (66)$$

The second of equations (66) has a geometric representation as a quadric surface — a cone.

The parametric forcing occurs when either $n \neq 1$ and $n \cong \omega_* 1$, $n = 1$ and $n \neq \omega_* 1$, or $n = 1 \cong \omega_* 1$. Consider the last most complex case of resonance. Let the following connections occur:

$$1^2 \omega_*^2 - 1 = a', \quad \lambda_* = \epsilon^2 \lambda_1, \quad (67)$$

where $a' = \epsilon^2 a$. Then, by proceeding as in the previous case one obtains

$$\delta_*^2 / (\omega_* / 2) - \frac{A^2}{(2/3\xi)} = 2(\mu_* - a'). \quad (68)$$

Equation (68) is the equation of hyperbolic paraboloid.

1.6.3. Concluding remarks

An analysis of the Hopf bifurcation in nonstationary non-linear systems exemplified by the Mathieu-Duffing oscillator has been presented. Considerations have been limited to determining the parameter-frequency relations and the dependence of the bifurcation parameter on the other parameters of the systems. The method of development into series of the perturbation parameter connected with the vibration amplitude (on the assumption that it is a small quantity), and the method of harmonic balancing have been employed. Cases with and without external force have been considered. For the latter case, with the vibration far from resonance, the vibration amplitude A , the amplitude of the excitation force λ_* and the frequency δ , create a quadric surface, which is a cone. On the other hand, for vibrations near resonance, the frequency δ_* , the vibration amplitude A , and the parametric excitation factor μ_* create a surface of the second degree, which is a hyperbolic paraboloid.

Chapter 2

2. BIFURCATION AND CHAOS: NUMERICAL METHOD BASED ON SOLVING BOUNDARY VALUE PROBLEM

2.1. Introduction

Examples of chaotic behaviour in deterministic nonlinear oscillators show that our knowledge cannot explain all the possible complex dynamics of even simple nonlinear systems [27-35]. One can also expect in complicated dynamical systems something more than strange attractors [36]. It is well known that the qualitative change in the behaviour of the investigated dynamical system with the accompanying change of its one or few parameters is due to bifurcations. The standard classical methods based on solving the initial value problem are not efficient enough to give a general structure of the bifurcations in a parameter space. For instance, they do not allow for the accurate calculation of the bifurcation points and for the calculation of unstable (invisible) attractors.

This chapter concentrates on the systematical approach to trace the behaviour of the system by varying one (or more) chosen parameters in autonomous and nonautonomous systems. The presented approach is based on solving a boundary value problem using the shooting method and develops earlier Brummundt's works [37,38], where the Urabe [39,40] method was used. Similar strategy of calculating branching points and new bifurcating branches of solutions is widely described by Seydel [41-44].

2.2. Gradual and Sudden Transition to Chaos

2.2.1. Method and results

A Duffing-type unsymmetric oscillator governed by the equation

$$\ddot{y} + c\dot{y} + y^3 = q + F\cos\omega t. \quad (69)$$

is reconsidered. This oscillator was investigated by Ueda [45]. He presented strange attractors for $\omega = 1$, $F = 0.16$, $c = 0.15$, $q = 0.03$ and $q = 0.045$. These results were obtained using the numerical method based on solving the initial value problem. Also Szemplinska-Stupnicka and Bajkowski [46] examined the behaviour of this oscillator by means of an approximate analytical method. The results obtained by them were then verified by the computer simulation analysis.

Here local bifurcations of periodic orbits in the q parameter line are investigated for the following fixed values of parameters: $c = 0.15$, $\omega = 1.0$, $F = 0.16$. The flow of (69) is strictly contracted, which implies only that the saddle-node or the period doubling local bifurcation exists. Hopf bifurcations are excluded. In order to investigate the k -subharmonic solution, the integration interval of (69) is transformed to $2\pi k$ length. By the use of the shooting method it is then possible to solve the following boundary value problem and to obtain multipliers from equations

$$\begin{aligned} M_i^{(m)}(y_1, y_2, q) - y_i &= 0, \\ \chi[DM^{(m)}(y_1, y_2, q), \delta] &= 0, \end{aligned} \quad (70)$$

$$y_1 = y, y_2 = \dot{y}, i = 1, 2.$$

Above, $M_i^{(m)}$ denotes the i -th component of the stroboscopic phase portrait (Poincaré map) of (69) with nondimensional time $\tau = \omega t$. χ is the characteristic polynomial whose eigenvalues δ_i are the sought multipliers of the $2\pi k$ periodic Flo-

quet matrix. Calculations were interrupted at the m -th step, only if the following norm $\|M_i^{(m)} - y_i\| = \sum_1^2 |M_i^{(m)} - y_i|^2 \leq 10^{-6}$. The multipliers are either real or complex conjugates. We will trace the movement of those multipliers which accompany the change of q .

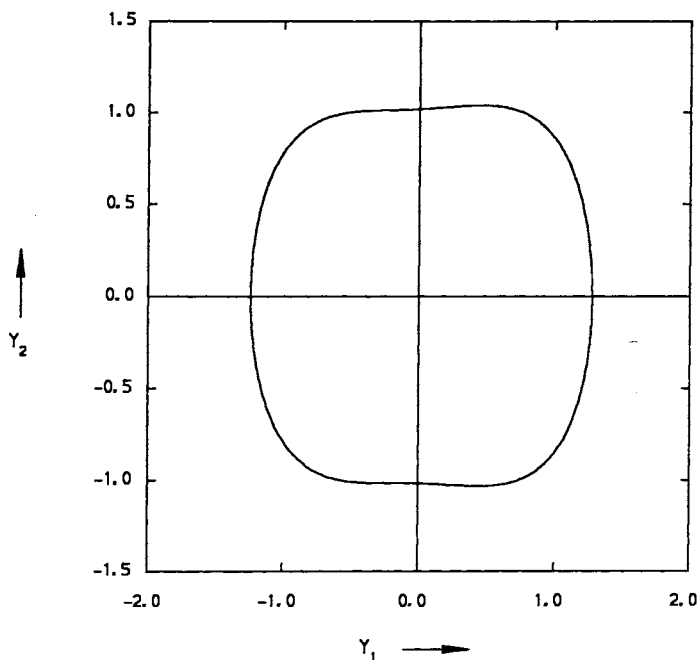


Fig. 1. Phase portrait of the large periodic orbit for $q = 0.03$.

The main resonance solution is presented in Fig. 1. As is seen from Table 1, the increase of q has almost no effect on the behaviour of the multipliers. However, for the considered interval of q , another solution exists (see Table 2). For $q \in (0.042385; 0.03913)$ this is the $1/2$ subharmonic solution. There are two possible routes to chaos for this solution (small orbit). The violent burst into a strange chaotic

Table 1. The main resonance solution found using tracing algorithm and corresponding multipliers.

q	y	$\dot{y}$	Multipliers	
			1	2
0.03	1.1406	0.65757	0.21; 0.83	0.21; -0.83
0.035	1.1432	0.66083	0.22; 0.82	0.22; -0.82
0.04	1.1458	0.66408	0.21; 0.83	0.21; -0.83
0.045	1.1483	0.66733	0.21; 0.83	0.21; -0.83
0.05	1.1509	0.67057	0.21; 0.83	0.21; -0.83

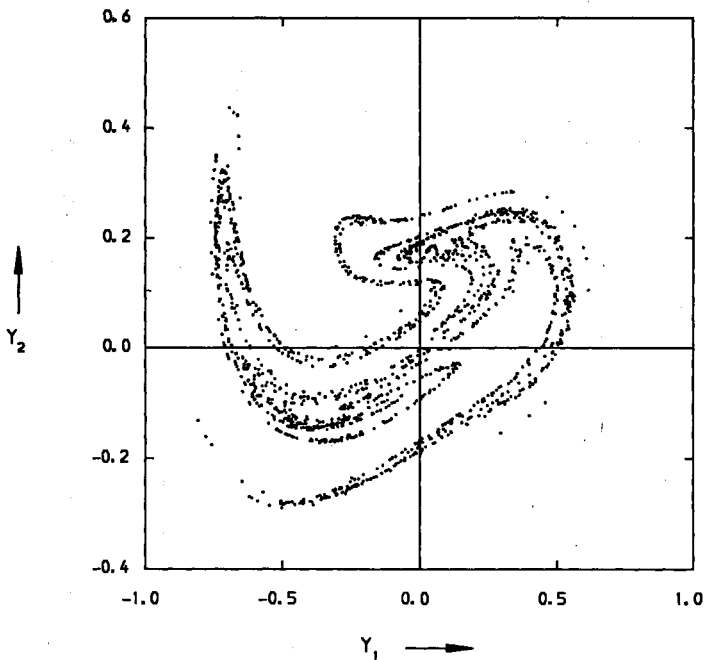


Fig. 2. Strange chaotic attractor obtained for $q = 0.043$
(one multiplier crosses the unit circle at +1).

Table 2. Bifurcation structure of the small periodic orbit and corresponding eigenvalues-example numerical results. Asterisks are coded bifurcation points.

q	y	$\dot{y}$	Multipliers		Period
			1	2	
0.042385*	-0.26362	-0.13127	1.0; 0.0	0.54; 0.0	4 π
0.042382	-0.26156	-0.13080	0.78; 0.0	0.09; 0.0	
0.042380	-0.26071	-0.13061	0.72; 0.13	0.72; -0.13	
0.042000	-0.23492	-0.12428	0.27; 0.67	0.27; -0.67	
0.041500	-0.22024	-0.12033	0.00; 0.73	0.00; -0.73	
0.040000	-0.19277	-0.11227	-0.54; 0.49	-0.54; -0.49	
0.039300	-0.18314	-0.10923	-0.65; 0.00	-0.83; 0.0	
0.039190*	-0.18173	-0.10877	-0.54; 0.00	-1.0 ; 0.0	
0.039150	-0.18966	-0.10992	0.33; 0.0	0.87; 0.0	8 π
0.039110	-0.19487	-0.11066	0.52; 0.10	0.52; -0.10	
0.039000	-0.20149	-0.11156	0.37; 0.38	0.37; -0.38	
0.038300	-0.22495	-0.11436	-0.70; 0.0	-0.40; 0.0	
0.03822*	-0.226810	-0.11455	-1.0 ; 0.0	-0.27; 0.0	
0.03815	-0.235540	-0.11602	-0.01; 0.24	-0.01; -0.24	16 π
0.03808	-0.214580	-0.11210	-0.28; 0.32	-0.28; -0.32	
0.03803	-0.21257	-0.11165	-0.23; 0.0	-0.88; 0.0	
0.038029*	-0.21253	-0.11164	-0.08; 0.0	-1.0 ; 0.0	