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Classical and Quantum Dynamics of Constrained Hamiltonian Systems

Heinz J Rothe | Klaus D Rothe

Classical and Quantum Dynamics of Constrained Hamiltonian Systems

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To Renate and our children
Christine, Stefan and Laura

Heinz Rothe

To Neusa Maria,
and our son Thomas

Klaus Rothe

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Classical and Quantum Dynamics of Constrained Hamiltonian Systems

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Preface

Since the pioneering work of Bergmann in 1949, it has been understood that there exists a direct connection between local symmetries of a Lagrangian and the existence of constraints contained in the Euler-Lagrange equations of motion. It was Dirac who subsequently discussed the Hamiltonian dynamics of constrained systems in a systematic way. Since then numerous papers, with applications to a wide range of problems, have been written on this subject. It is remarkable that even after more than half a century, this subject still raises questions to be answered.

This book is the result of lectures delivered by the authors at the University of Heidelberg. It is intended as an introduction to this interesting field of research, starting from the early work of Dirac up to more recent work, such as the Field-Antifield formalism of Batalin and Vilkovisky. Also included is a brief discussion of gauge anomalies within this formalism. Here we have restricted ourselves to the essentials, and have illustrated the ideas in terms of a simple example, the chiral Schwinger model.

Great emphasis is placed on discussing the subject of constrained Hamiltonian systems in a transparent way without becoming too technical. Proofs, as well as examples, are discussed in great detail, at the risk of being sometimes pedantic. Each example has been chosen to illustrate some important point. The book should thus hopefully provide a good basis for master degree and Ph.D students.

The book is divided into two parts: part one, involving chapters 1 through 8, deals with the classical dynamics of constrained Hamiltonian systems, while the second part is devoted to their quantization. Here it is assumed that the reader is familiar with the Feynman path integral approach to quantization.

Notation

The Einstein summation convention is used for repeated upper and lower indices. Except for Lorentz space-time indices, we have placed indices, labeling the degrees of freedom, at our convenience, so as to keep the expressions most transparent. We have followed to a large extent the notation given below, although in particular cases - such as in examples - we have found it convenient to follow a simpler notation.

Notation

ϕ_α : primary constraints

$\phi_{\alpha_1}^{(1)}$: first class primary constraints

$\phi_{\alpha_2}^{(2)}$: second class primary constraints

φ_a : secondary constraints

$\varphi_{a_1}^{(1)}$: first class secondary constraints

$\varphi_{a_2}^{(2)}$: second class secondary constraints

$\Omega_{A_1}^{(1)}$: complete set of first class constraints

$\Omega_{A_2}^{(2)}$: complete set of second class constraints

Γ_P : subspace defined by primary constraints

Γ : subspace defined by all constraints

Γ^* : subspace defined by the constraints and gauge conditions

H_0 : canonical Hamiltonian on the primary constrained surface Γ_P

H : Hamiltonian weakly equivalent to H_0

$\tilde{H}$: BFT embedded Hamiltonian

H_E : extended Hamiltonian

$\tilde{H}_E$ fully extended Hamiltonian, including the gauge conditions

H^* : Hamiltonian on the reduced phase-space

Q_B : generator of BRST transformations: $\delta_B \mathcal{F} = \epsilon \{Q_B, \mathcal{F}\} = \{\mathcal{F}, Q_B\} \epsilon$

s : operator inducing “left” BRST transformations: $s\mathcal{F} = \{Q_B, \mathcal{F}\}$

$\hat{s}$: operator inducing “right” BRST transformations $\hat{s}\mathcal{F} = \{\mathcal{F}, Q_B\}$

$\bigvee s$: operator inducing “right” BRST transformations on Lagrangian level

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Chapter 1

Introduction

Elementary particle physicists are well familiar with Quantum Electrodynamics (QED) and Quantum Chromodynamics (QCD), the theories describing the electromagnetic and strong interactions. These theories are based on Lagrangians possessing local (gauge) symmetries. The weak interactions are also based on a gauge theory, but whose gauge invariance is spontaneously broken. In fact, all theories of the fundamental interactions are gauge theories. This makes the study of systems possessing local symmetries so important. As is well known, the quantization of such systems is endowed with special problems, since their classical action is invariant along gauge orbits. On the level of the equations of motion this implies the existence of an infinite number of solutions connected by local (gauge) transformations, which are physically completely equivalent. Several methods of quantization, such as the canonical Gupta-Bleuler formalism, or the functional path integral approach, which makes use of the so-called Faddeev-Popov trick, are well known to theoretical elementary particle physicists. These methods are described in any textbook on Quantum Field Theory.

The question then arises whether there exists a fundamental feature which is common to all gauge theories, and which inhibits their straightforward quantization. The answer is “yes”; they all belong to the so-called class of singular systems, characterized by the property, that the Euler-Lagrange equations of motion include equations which represent constraints among the coordinates and the velocities, and which on Hamiltonian level manifest themselves as so-called “first class” constraints among canonical variables. These constraints differ from those familiar from mechanics in that they are part of the equations of motion, and are not imposed from the outside. Although gauge theories are certainly the most important examples of singular Lagrangian systems, they

are not the only ones belonging to this class. Even the dynamics of the free Dirac field is that of a constrained system.

A deeper understanding of the quantization of gauge theories requires one to go back to the roots of the quantization problems associated with singular Lagrangians. The basis for understanding such systems was laid down by Bergmann [Bergmann 1949, Anderson 1951], and by Dirac [Dirac 1950/1964]. Since then an enormous amount of papers have been written about constrained Hamiltonian systems.

This book is mainly devoted to the study of the classical and quantum dynamics of theories where the Lagrangian exhibits local symmetries. Its purpose is to introduce the reader to this interesting field of constrained dynamical systems, and to provide him with the tools for quantizing such systems. This will require first of all an extensive discussion of the classical Hamiltonian formulation of singular systems. Our language will be kept as simple as possible, and the concepts we shall introduce will always be illustrated by examples in a transparent way as possible. The classical discussion is contained in chapters 2 through 8. The remaining chapters are devoted to the quantization of singular systems.

The book is organized as follows. In chapter 2 we first discuss what is meant by a singular Lagrangian system, and show that systems with a local (gauge) symmetry fall into this class. Local symmetries of the action are not always easily detected. It is however crucial to unravel them, since their knowledge is required for the quantization of such systems. Although their quantization is carried out on Hamiltonian level, it is instructive to first present a systematic method for detecting local symmetries on Lagrangian level. This is the main objective of chapter 2, where we shall show that singular systems lead to so-called “gauge identities”, from which the coordinate transformations, leaving the action invariant, can be extracted.

In chapter 3 we then proceed to the classical Hamiltonian formulation based on the pioneering work of Dirac. It is within this framework that Dirac has provided a systematic classification of the constraints into *primary*, *secondary*, *first* and *second class*, where the first class constraints play a central role as generators of the gauge symmetries.

Chapter 4 is devoted to an alternative method for handling singular systems: the symplectic approach. This is a first order Lagrangian approach to singular systems, and we shall study in detail the equations of motion and their solution. The symplectic approach has been used by Faddeev and Jackiw [Faddeev 1988] to present a new method for handling constrained systems, which, in principle, avoids the concepts of primary constraints (which have no analog on the Lagrangian level), and in fact dispenses of the entire classification of the constraints into primary, secondary, first and second class. This method is however of limited applicability.

The Hamiltonian approach for detecting local symmetries is the phase-space counterpart to our discussion in chapter 2, and is the subject of chapter 5. We shall elucidate the problem from several points of view and are thereby led to the so-called “Dirac conjecture”, which has been the subject of numerous papers.

The Dirac conjecture will be discussed in detail in chapter 6, where we make use of the Lagrangian methods discussed chapter 2 to study the local symmetries of the so-called total action, whose variation yields the Hamilton equations of motion proposed by Dirac. This chapter clarifies in which sense the Dirac conjecture can be shown to hold. Interpreted appropriately, this conjecture is shown to hold for various examples cited in the literature, including those which have been claimed to be counterexamples to Dirac’s conjecture.

Chapter 7 deals with second class systems whose classical dynamics is beautifully described in terms of Dirac brackets, but whose quantization may pose special problems. For this reason it is useful to embed such systems into a gauge theory, by introducing extra degrees of freedom. The embedded theories can then be quantized using the formalism described in chapter 11. This embedding program, whose systematic implementation was given by Batalin, Fradkin and Tyutin, [Batalin 1987/1991] is the central subject of this chapter. We shall present several examples, and in particular use this formalism to demonstrate the equivalence of the “self dual model” in three space-time dimensions, with the Maxwell-Chern-Simons theory within its gauge invariant sector.

Chapter 8 deals with the Hamilton-Jacobi formulation of the classical dynamics of constrained systems. While this formulation turns out to be straightforward for so-called first class systems, this is not the case for systems involving second class constraints. We shall discuss the reason for this, and present methods for circumventing the barriers imposed by the second class constraints. In particular, the embedding procedure described in chapter 7 will allow one to arrive at a Hamilton-Jacobi like formulation, as we shall demonstrate for several examples.

In the following chapters we then turn to the problem of quantizing systems with first and second class constraints. Chapter 9 will deal with purely second class systems which allow an operator quantization, based on equations of motion formulated using the notion of an extended Hamiltonian. Second class systems also include the case of gauge theories in a non-dynamical gauge.

In chapter 10 we then discuss in detail the quantization of second class systems using functional methods, and point out the difficulties one may be confronted with. These difficulties may be circumvented by first embedding the system into a gauge theory, as was described in chapter 7.

The functional quantization of first class systems in dynamical gauges is the subject of chapter 11 and leads to the Hamiltonian formulation of Batalin,

Fradkin and Vilkovisky [Fradkin 1975/77, Batalin 1977], based on the so-called BRST (Becchi, Rouet, Stora, Tyutin) invariance of the Lagrangian action. We shall not follow the axiomatic approach of these authors, but shall motivate the BFV formalism by studying systems whose functional configuration space formulation, obtained using the Faddeev-Popov trick, is well established. We then obtain the corresponding phase-space representation of the partition function and cast it into the form given in [Batalin 1977]. The so-called BRST charge will play a central role as generator of a BRST symmetry. The freedom in the choice of gauge will be concentrated in a BRST exact term, and the BRST invariance of the partition function is guaranteed by the Fradkin-Vilkovisky principal theorem [Fradkin 1977, Batalin 1977]. Important ingredients in this formulation is the closure of the algebra of the first class constraints, as well as of these constraints with the Hamiltonian.

In the Hamiltonian formulation it is not obvious how a particular covariant gauge is to be implemented via a BRST exact contribution to the action. A corresponding configuration space formulation is therefore desirable. This is the subject of the last chapter and leads to the so-called “field-antifield” formulation of Batalin and Vilkovisky [Batalin 1981]. This quantization scheme has been presented by these authors in an axiomatic way. We shall actually derive their formulation for a restricted class of systems, starting from the well established phase-space formulation of Batalin, Fradkin and Vilkovisky. Within the field-antifield formulation the number of degrees of freedom is doubled, and the field-antifield action is the solution of the so-called “quantum master equation”. We will discuss only the simplest non-trivial case of irreducible first rank theories. These include the case of $SU(N)$ Yang-Mills theories.

We conclude chapter 12 with a brief discussion of how anomalous gauge theories - i.e., theories whose classical gauge invariance is broken by quantum effects due to the non-invariance of the functional integration measure - may be included in the field-antifield formalism. This subtle subject is outside the scope of this book. We shall therefore limit ourselves to illustrate the role played by the quantum master equation for the case of the chiral Schwinger model, which is anomalous and has been studied in great detail in the literature.

The book includes several appendices. In Appendix A we review some well known facts regarding the Ward and Bianchi identities following from the existence of a local symmetry of the action. This appendix is based on the pioneering work of P. G. Bergmann [Bergmann 1949], who has pointed out the connection between the existence of a local symmetry and a so-called “singular Lagrangian”. The work of Bergmann has been reviewed in [Costa 1988]. We shall follow closely the presentation given in this reference.

Appendices B and C deal with technical details concerning the construction of the BRST charge and Hamiltonian for first rank theories.

Appendix D contains a proof of the Fradkin-Vilkovisky Principal Theorem,

which provides the basis for the Batalin-Vilkovisky configuration space partition function in chapter 12.

Finally, Appendix E discusses the Fradkin-Vilkovisky path integral representation of the $SU(3)$ Yang-Mills partition function in the α -gauges.

Chapter 2

Singular Lagrangians and Local Symmetries

2.1 Introduction

Symmetries observed in nature play a central role in the construction of any theory describing physical phenomena. If the dynamics is described by a Lagrangian, then this Lagrangian must reflect such symmetries. Not all symmetries of an action can however be observed in nature. This is for example the case for the gauge symmetries of QED or QCD, where observables must be invariant under gauge transformations. Gauge theories fall under the class of constrained systems whose dynamics is derived from a so-called singular Lagrangian. Such theories are of particular interest, since all interactions in nature are believed to be gauge theories, or spontaneously broken versions thereof. As is well known, their quantization poses special problems.

Whether a Lagrangian is singular or not can in general be easily checked by looking for possible zero modes of the Hessian (to be defined below). The connection between a singular Hessian and local symmetries of a Lagrangian has been observed already a long time ago [Bergmann 1949]. Such symmetries lead to so-called Bianchi identities from which this connection can be deduced. The interested reader may consult Appendix A, where we have summarized some well known concepts about local symmetries, as well as consequences following from them. In this chapter we take the opposite point of view and deduce the local symmetries of a Lagrangian whose Hessian matrix is not invertible. In fact, unravelling these symmetries may not be an easy matter. But their explicit knowledge is important for formulating the corresponding quantum theory. It is therefore of great interest to have a systematic way for revealing them. In

the following we will present a purely Lagrangian approach to this important problem [Sudarshan 1974]. This will allow the reader to appreciate better the elegant Hamiltonian approach to be discussed in chapter 3, where we show that such symmetries are intimately related to the existence of so-called “first class constraints”. Both approaches have their merits. But the connection between the existence of local symmetries, and the existence of first class constraints as generators of these symmetries, is only manifest on Hamiltonian level.

On the Lagrangian level there exist several algorithms [Sudarshan 1974, Chaichian 1994, Shirzad 1998] for detecting the gauge symmetries of a Lagrangian. Unlike the Hamiltonian approach, they have the merit of generating directly the transformation laws in configuration space in terms of an independent set of functions parametrizing the symmetry transformations. Every one of these parameters is directly related to a so-called “gauge identity”, and their number equals the number of independent gauge identities.

2.2 Singular Lagrangians

Consider a system with a finite number of degrees of freedom, whose dynamics is governed by a Lagrangian $L(q, \dot{q})$, where q and $\dot{q}$ stand collectively for the coordinates q_i and velocities $\dot{q}_i$ ($i = 1, \dots, N$). The associated Euler-Lagrange equations read,

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0. \quad (2.1)$$

From here it follows that

$$\frac{\partial^2 L}{\partial q_j \partial \dot{q}_i} \frac{dq_j}{dt} + \frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_i} \frac{d\dot{q}_j}{dt} - \frac{\partial L}{\partial q_i} = 0,$$

or

$$\sum_j W_{ij}(q, \dot{q}) \ddot{q}_j = \frac{\partial L}{\partial q_i} - \sum_j \frac{\partial^2 L}{\partial \dot{q}_i \partial q_j} \dot{q}_j,$$

where the matrix W is the “Hessian” defined by

$$W_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}. \quad (2.2)$$

If $\det W \neq 0$, we can solve the above equation for the accelerations in terms of $\dot{q}_i$ and q_i . On the other hand, if $\det W = 0$, then W is not invertible. In this case the Hamiltonian equations of motion will not take the standard form, and one speaks of a singular Lagrangian system.

The following two examples illustrate some features which we shall subsequently encounter in our more general discussion of local symmetries.

i) *Example 1*

Consider the Lagrangian

$$L(q, \dot{q}) = \frac{1}{2}\dot{x}^2 + \dot{x}y + \frac{1}{2}(x - y)^2 \quad (2.3)$$

and the associated action

$$S[q] = \int_{t_1}^{t_2} dt L(q(t), \dot{q}(t)) , \quad (2.4)$$

where $q = (x, y)$. This Lagrangian is singular in the above sense, since it does not involve $\dot{y}$. Hence, we claim, it will possess a local symmetry. In this example this symmetry can be found by inspection.

Consider a variation $\delta x = \epsilon_1(t)$, $\delta y = \epsilon_2(t)$. Under this variation the change induced in the Lagrangian can be written in the form

$$\delta L = (\dot{x} + y - x)(\dot{\epsilon}_1 - \epsilon_1 + \epsilon_2) + \frac{d}{dt}(x\epsilon_1) .$$

Hence the corresponding action (2.4) is left invariant under this transformation if

$$\epsilon_2 = \epsilon_1 - \dot{\epsilon}_1 , \quad (2.5)$$

with $\epsilon_1(t)$ vanishing for $t = t_1$ and $t = t_2$. Thus for an arbitrary choice of $\epsilon_1(t)$, subject to this condition, ϵ_2 is given in terms of ϵ_1 and its *first* derivative.

ii) *Example 2*

Consider next the Lagrangian [Henneaux 1990a]

$$L(q, \dot{q}) = \frac{1}{2}(\dot{q}_2 - e^{q_1})^2 + \frac{1}{2}(\dot{q}_3 - q_2)^2 . \quad (2.6)$$

Again this Lagrangian is singular and therefore - we claim - will exhibit a local symmetry. Thus consider a variation $\delta q_i = \epsilon_i(t)$. The corresponding change induced in the Lagrangian is given by

$$\delta L = (\dot{q}_2 - e^{q_1})(\dot{\epsilon}_2 - e^{q_1}\epsilon_1) + (\dot{q}_3 - q_2)(\dot{\epsilon}_3 - \epsilon_2) .$$

Hence the Lagrangian (2.6) is invariant if

$$\begin{aligned} \epsilon_3(t) &= \alpha(t) , \\ \epsilon_2(t) &= \frac{d}{dt}\alpha(t) , \end{aligned}$$

$$\epsilon_1(t) = e^{-q_1(t)} \frac{d^2}{dt^2} \alpha(t) , \quad (2.7)$$

where $\alpha(t)$ is an arbitrary function of t . Notice that, as in our previous example, the symmetry transformation only involves one arbitrary function. Furthermore, in this example, the variation δq_1 not only involves the arbitrary function $\alpha(t)$, but also depends implicitly on t via the coordinate q_1 .

In the following section we present a general algorithm which allows one to determine the variations δq_i which leave an action invariant.

2.3 Algorithm for detecting local symmetries on Lagrangian level

On the Lagrangian level there exists a well-known algorithm [Sudarshan 1974] for detecting all gauge symmetries of a Lagrangian.

Consider an infinitesimal variation of the coordinates and velocities

$$\begin{aligned} q_i(t) &\rightarrow q_i(t) + \delta q_i(t) , \\ \dot{q}_i(t) &\rightarrow \dot{q}_i(t) + \frac{d}{dt} \delta q_i(t) , \end{aligned}$$

where $i = 1, \dots, N$, and use has been made of the property that $\delta \dot{q}_i = \frac{d}{dt} \delta q_i$. This transformation is a local symmetry of the Euler-Lagrange equations of motion if the action (2.4) is invariant under this transformation. This is the case if the variation of the Lagrangian is a total derivative, i.e., $\delta L = \frac{d}{dt} \delta F$, with δF vanishing at the upper and lower limit of integration. In computing the variation no use is made of the equations of motion.

Consider now a Lagrangian $L(q, \dot{q})$. Under the above infinitesimal transformation the change in the action (2.4) is given by

$$\delta S = - \int_{t_1}^{t_2} dt \sum_i E_i^{(0)}(q, \dot{q}, \ddot{q}) \delta q_i , \quad (2.8)$$

where $E_i^{(0)}$ is the Euler derivative

$$E_i^{(0)}(q, \dot{q}, \ddot{q}) = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} \quad (2.9)$$

and $\delta q_i(t_1) = \delta q_i(t_2) = 0$. On shell we have that

$$E_i^{(0)} = 0 , \quad i = 1, \dots, N \quad (\text{on shell}) . \quad (2.10)$$

Our aim is to determine the variations δq_i for which δS vanishes identically, i.e.,

$$\int_{t_1}^{t_2} dt \sum_i E_i^{(0)}(q, \dot{q}, \ddot{q}) \delta q_i \equiv 0 . \quad (2.11)$$

In general, $E_i^{(0)}$ is of the form

$$E_i^{(0)} = \sum_j W_{ij}^{(0)}(q, \dot{q}) \ddot{q}_j + K_i^{(0)}(q, \dot{q}) , \quad (2.12)$$

where $W_{ij}^{(0)}$ is the Hessian (2.2). For reasons which will become clear below, we have supplied the terms in (2.12) with a superscript “0” standing for “zeroth level”.

Now, as we have just seen, for singular Lagrangians the Lagrange equations of motion cannot be solved for all of the accelerations. If R_W is the rank of the Hessian, then there exist $N - R_W$ constraints in the theory, which on Lagrangian level present themselves as relations between coordinates q_i and velocities $\dot{q}_i$. Thus for an $N \times N$ Hessian $W_{ij}^{(0)}$ of rank R_W , there exist $N - R_W$ independent left (or right), zero-mode eigenvectors $\vec{w}^{(0;k)}$ satisfying

$$\sum_{i=1}^N w_i^{(0;k)}(q, \dot{q}) W_{ij}^{(0)}(q, \dot{q}) = 0 ; \quad k = 1, \dots, N - R_W .$$

It then follows from (2.12) that the functions defined by

$$\Phi^{(0;k)} := \sum_{i=1}^N w_i^{(0;k)}(q, \dot{q}) E_i^{(0)}(q, \dot{q}, \ddot{q})$$

only depend on the coordinates and velocities, and vanish on the subspace of physical trajectories, i.e.,

$$\Phi^{(0;k)}(q, \dot{q}) = 0 ; \quad k = 1, \dots, N - R_W \quad (\text{on shell}) .$$

We refer to them as the zero-generation constraints.¹ Not all of the $\Phi^{(0;k)}$ may, however, be linearly independent. In this case one can find linear combinations of the zero mode eigenvectors, $v_i^{(0;n_0)} = \sum_k c_k^{(n_0)} w_i^{(0;k)}$, such that we have identically

$$G^{(0;n_0)} := \vec{v}^{(0;n_0)} \cdot \vec{E}^{(0)} \equiv 0 ; \quad n_0 = 1, \dots, N_0 . \quad (2.13)$$

We call them “gauge identities” for reasons which will become clear soon. These gauge identities also hold off shell. As an immediate consequence we have that

¹For simplicity we shall suppress in the following the explicit dependence of the constraints and zero modes on q and $\dot{q}$.

any variation of the form $\delta q_i = \sum_{n_0} \epsilon_{n_0}(t) v_i^{(0;n_0)}$ will leave the action invariant (cf. (2.11)). The remaining “zero-generation” zero modes, which we denote by $\vec{u}^{(0;\bar{n}_0)}$, then lead to genuine constraints, depending only on the coordinates and velocities, and vanishing on shell,

$$\phi^{(0;\bar{n}_0)} = 0 \quad (\text{on shell}) ,$$

where

$$\phi^{(0;\bar{n}_0)} = \vec{u}^{(0;\bar{n}_0)} \cdot \vec{E}^{(0)} ; \quad \bar{n}_0 = 1, \dots, \bar{N}_0 . \quad (2.14)$$

The algorithm now proceeds as follows. We separate the gauge identities (2.13) from the non-trivial constraints (2.14), and list them separately in a table, since they will be of direct relevance for determining the local symmetry transformations. Next we look for possible additional constraints by searching for further functions of the coordinates and velocities which vanish on the subspace of physical trajectories. To this effect consider the following $N + \bar{N}_0$ component vector constructed from $\vec{E}^{(0)}$ and the time derivative of the constraints (2.14):

$$\left(E_{i_1}^{(1)} \right) := \begin{pmatrix} \vec{E}^{(0)} \\ \frac{d}{dt}(\vec{u}^{(0;1)} \cdot \vec{E}^{(0)}) \\ \vdots \\ \frac{d}{dt}(\vec{u}^{(0;\bar{N}_0)} \cdot \vec{E}^{(0)}) \end{pmatrix} = \begin{pmatrix} \vec{E}^{(0)} \\ \frac{d}{dt} \vec{\phi}^{(0)} \end{pmatrix} , \quad (2.15)$$

where $\vec{\phi}^{(0)}$ is an $\bar{N}_0$ component column vector with components $\phi^{(0;\bar{n}_0)}$, ($\bar{n}_0 = 1, \dots, \bar{N}_0$). Since the constraints $\phi^{(0;\bar{n}_0)} = 0$ hold for all times, it follows that $\vec{E}^{(1)} = 0$ on shell.

Because the constraints are only functions of q and $\dot{q}$, the components $E_{i_1}^{(1)}$ can again be written in a form paralleling (2.12),

$$E_{i_1}^{(1)} = \sum_{j=1}^N W_{i_1 j}^{(1)}(q, \dot{q}) \ddot{q}_j + K_{i_1}^{(1)}(q, \dot{q}) . \quad (2.16)$$

Here $\mathbf{W}^{(1)}$ is now the “level 1” $(N + \bar{N}_0) \times N$ matrix

$$\left(W_{i_1 i}^{(1)} \right) := \begin{pmatrix} \mathbf{W}^{(0)} \\ \vec{\nabla}_{\dot{q}}(\vec{u}^{(0;1)} \cdot \vec{E}^{(0)}) \\ \vdots \\ \vec{\nabla}_{\dot{q}}(\vec{u}^{(0;\bar{N}_0)} \cdot \vec{E}^{(0)}) \end{pmatrix} ,$$

where $\vec{\nabla}_{\dot{q}}$ is a *row* vector with components labeled by i ($i = 1, \dots, N$). The $(N + \bar{N}_0)$ component vector $\vec{K}^{(1)}$ is given by

$$\left(K_{i_1}^{(1)} \right) := \begin{pmatrix} \vec{K}^{(0)} \\ \frac{\partial}{\partial q_j} (\vec{u}^{(0;1)} \cdot \vec{E}^{(0)}) \dot{q}_j \\ \vdots \\ \frac{\partial}{\partial q_j} (\vec{u}^{(0;\bar{N}_0)} \cdot \vec{E}^{(0)}) \dot{q}_j \end{pmatrix},$$

where a summation over j is understood.

We next look again for (left) zero-modes of $\mathbf{W}^{(1)}$. These zero modes include of course those of the previous level, augmented by an appropriate number of zeroes. They just reproduce the previous constraints and are therefore not considered. The remaining zero modes (if they exist), when contracted with $\vec{E}^{(1)}$, lead to expressions at “level 1” which can either be written as a linear combination of the previous constraints (we label them by n_1), in which case we are led to new gauge identities at level 1,

$$G^{(1;n_1)} := \vec{v}^{(1;n_1)} \cdot \vec{E}^{(1)} - \sum_{\bar{n}_0=1}^{\bar{N}_0} M_{n_1, \bar{n}_0}^{(1,0)} (\vec{u}^{(0;\bar{n}_0)} \cdot \vec{E}^{(0)}) \equiv 0; \quad n_1 = 1, \dots, N_1, \quad (2.17)$$

or else, represent genuine new constraints at level 1:

$$\phi^{(1;\bar{n}_1)} := \vec{u}^{(1;\bar{n}_1)} \cdot \vec{E}^{(1)} = 0; \quad \bar{n}_1 = 1, \dots, \bar{N}_1 \quad (\text{on shell}). \quad (2.18)$$

We now adjoin the new gauge identities (2.17) to the previous identities in our table. With the remaining new constraints (2.18) we proceed as before, adjoining their time derivative to (2.15), and construct $W_{i_2 i}^{(2)}$ as well as $K_{i_2}^{(2)}$. The iterative process will terminate at some level M if either *i*) there are no further zero modes, or *ii*) the constraints generated are linear combinations of the previous ones, and hence lead to gauge identities only. At this point our algorithm has unravelled all the constraints of the Euler-Lagrange equations of motion.

We now look for the maximal set of *linearly independent* gauge identities generated by the algorithm. At each level ℓ , the elements of this set are of the form

$$G^{(0,n_0)} := \vec{v}^{(0;n_0)} \cdot \vec{E}^{(0)} \equiv 0, \quad (2.19)$$

$$G^{(\ell,n_\ell)} := \vec{v}^{(\ell;n_\ell)} \cdot \vec{E}^{(\ell)} - \sum_{\ell'=0}^{\ell-1} \sum_{\bar{n}_{\ell'}=1}^{\bar{N}_{\ell'}} M_{n_\ell, \bar{n}_{\ell'}}^{(\ell,\ell')} \phi^{(\ell';\bar{n}_{\ell'})} \equiv 0, \quad (2.20)$$

where $\ell = 1, \dots, N_\ell$, and where the $M_{n_\ell, n_{\ell'}}^{(\ell, \ell')}$ are only functions of q and $\dot{q}$, and

$$\phi^{(\ell; \bar{n}_\ell)} = \bar{u}^{(\ell; \bar{n}_\ell)} \cdot \vec{E}^{(\ell)} ; \quad (\bar{n}_\ell = 1, \dots, \bar{N}_\ell) \quad (2.21)$$

are the genuine constraints (depending only on q and $\dot{q}$) generated by the algorithm at level ℓ . On the other hand, $\vec{E}^{(\ell)}$ is given by the $N + \bar{N}_0 + \dots + \bar{N}_{\ell-1}$ component vector

$$\vec{E}^{(\ell)} := \begin{pmatrix} \vec{E}^{(0)} \\ \frac{d}{dt} \vec{\phi}^{(0)} \\ \vdots \\ \frac{d}{dt} \vec{\phi}^{(\ell-1)} \end{pmatrix}, \quad (2.22)$$

where $\vec{\phi}^{(\ell)}$ is a column vector with $\bar{N}_\ell$ components $\phi^{(\ell; \bar{n}_\ell)}$. Because of (2.21) and (2.22), we can proceed iteratively to express each of the constraints $\phi^{(\ell; \bar{n}_\ell)}$ in terms of $\phi^{(0; \bar{n}_0)} = \bar{u}^{(0; \bar{n}_0)} \cdot \vec{E}^{(0)}$ and time derivatives thereof. One then readily verifies that these constraints are of the form

$$\phi^{(\ell; \bar{n}_\ell)} = \sum_i \sum_{m=0}^{\ell} \eta_{mi}^{(\ell; \bar{n}_\ell)} \frac{d^m}{dt^m} E_i^{(0)} = 0.$$

From here, and the expression (2.22), we conclude that the identities (2.19) are of the following general form

$$G^{(\ell; n_\ell)} = \sum_i \sum_{m=0}^{\ell} \rho_{mi}^{(\ell; n_\ell)} \frac{d^m}{dt^m} E_i^{(0)} \equiv 0,$$

where $\rho_{mi}^{(\ell; n_\ell)}$ are in general functions of the coordinates and time derivatives thereof. It therefore also follows that

$$\sum_{\ell=0}^M \sum_{n_\ell=1}^{N_\ell} \epsilon^{(\ell; n_\ell)}(t) G^{(\ell; n_\ell)} \equiv 0,$$

where the $\epsilon^{(\ell; n_\ell)}(t)$ are arbitrary functions of time. This identity can be written in the form

$$\sum_i \delta q_i E_i^{(0)} - \frac{d}{dt} F \equiv 0, \quad (2.23)$$

where

$$\delta q_i = \sum_{\ell=0}^M \sum_{n_\ell=1}^{N_\ell} \sum_{m=0}^{\ell} (-1)^m \frac{d^m}{dt^m} \left(\rho_{mi}^{(\ell; n_\ell)} \epsilon^{(\ell; n_\ell)}(t) \right), \quad (2.24)$$

and F is a function of the coordinates and their time derivatives, as well as a linear function of the parameters and their derivatives up to order $M - 1$, where $\ell = M$ is the highest level. Note that the arbitrary functions $\epsilon^{(\ell; n_\ell)}(t)$ are labeled by ℓ , and n_ℓ . Hence the number of independent functions equals the number of gauge identities generated by the algorithm. Let us collect the indices ℓ and n_ℓ into a single index a . Then (2.24) reads

$$\delta q_i = \sum_{m,a} 1)^m \frac{d^m}{dt^m} \left(\rho_{mi}^{(a)} \epsilon^{(a)}(t) \right). \quad (2.25)$$

Now, integrating (2.23) in time from t_1 to t_2 , and using the fact that the total derivative will not contribute if the $\epsilon^{(\ell; \bar{n}_\ell)}(t)$ and their derivatives vanish for $t = t_1$ and $t = t_2$ (implying the vanishing of δq_i at the endpoints in (2.11)), we conclude that the variation (2.25) is a symmetry of the action, and hence of the Lagrange equations of motion. It should be noted, however, that a term of the form

$$\Delta q_i = \sum_j a_{ij} E_j^{(0)},$$

with an arbitrary antisymmetric tensor a_{ij} , can always be added to (2.25), and it will remain a symmetry transformation, since $E_i^{(0)} \Delta q_i \equiv 0$. These variations, which vanish on shell, are called “trivial gauge transformations”. They have no physical significance.

In the following we consider some examples which illustrate the above iterative methods for generating the gauge identities. Since in the first two examples there exists only one zero mode of the type u or v at each level, we will omit the indexes n_ℓ and $\bar{n}_\ell$ labeling such zero modes at level ℓ .

2.4 Examples

Example 1

Consider once more the Lagrangian (2.3). Let $x_1 = x$ and $x_2 = y$. Then

$$E_i^{(0)} = \sum_j W_{ij}^{(0)} \ddot{x}_j + K_i^{(0)},$$

where

$$\mathbf{W}^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\mathbf{K}^{(0)} = \begin{pmatrix} \dot{x}_2 + x_2 - x_1 \\ x_1 - x_2 - \dot{x}_1 \end{pmatrix}.$$

There exists only a single left zero mode, $\vec{u}^{(0)} = (0, 1)$. Its contraction with $\vec{E}^{(0)}$ does not vanish identically, but generates a constraint:

$$\phi^{(0)} \equiv \vec{u}^{(0)} \cdot \vec{E}^{(0)} = x_1 - x_2 - \dot{x}_1 = 0 \quad (\text{on shell}) . \quad (2.26)$$

Hence at level zero we have one constraint and no gauge identity. We next construct $\vec{E}^{(1)}$:

$$\vec{E}^{(1)} = \left(\begin{array}{c} \vec{E}^{(0)} \\ \frac{d}{dt} (\vec{u}^{(0)} \cdot \vec{E}^{(0)}) \end{array} \right) = \left(\begin{array}{c} \vec{E}^{(0)} \\ \frac{d}{dt} E_2^{(0)} \end{array} \right) , \quad (2.27)$$

or, written out explicitly

$$\vec{E}^{(1)} = \left(\begin{array}{c} \ddot{x}_1 + \dot{x}_2 - x_1 + x_2 \\ x_1 - x_2 - \dot{x}_1 \\ \dot{x}_1 - \dot{x}_2 - \ddot{x}_1 \end{array} \right) .$$

$\vec{E}^{(1)}$ can again be written in the form

$$E_{i_1}^{(1)} = \sum_j W_{i_1 j}^{(1)} \ddot{x}_j + K_{i_1}^{(1)} ,$$

where the 3×2 matrix $\mathbf{W}^{(1)}$ is given by

$$\mathbf{W}^{(1)} = \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \\ -1 & 0 \end{array} \right) ,$$

and

$$\mathbf{K}^{(1)} = \left(\begin{array}{c} \dot{x}_2 + x_2 - x_1 \\ x_1 - x_2 - \dot{x}_1 \\ \dot{x}_1 - \dot{x}_2 \end{array} \right) .$$

The left zero modes of $\mathbf{W}^{(1)}$ are given by $(0, 1, 0)$ and $(1, 0, 1)$. The first zero mode is just the previous zero mode augmented by one zero, and reproduces the previous constraint. We need not consider it again. The second zero mode reproduces the (negative) of the constraint (2.26), and is thus of the type v :

$$\vec{v}^{(1)} \cdot \vec{E}^{(1)} \equiv -\vec{u}^{(0)} \cdot \vec{E}^{(0)} ,$$

where $\vec{v}^{(1)} = (1, 0, 1)$. This leads to the gauge identity

$$G^{(1)} = \vec{v}^{(1)} \cdot \vec{E}^{(1)} + \vec{u}^{(0)} \cdot \vec{E}^{(0)} \equiv 0$$

at level 1. Noting from (2.27) that

$$\vec{v}^{(1)} \cdot \vec{E}^{(1)} = E_1^{(0)} + \frac{d}{dt} E_2^{(0)} ,$$

the above identity takes the form

$$E_1^{(0)} + \frac{d}{dt} E_2^{(0)} + E_2^{(0)} \equiv 0 .$$

Multiplying this expression by an arbitrary function of time, $\epsilon(t)$, we are led to

$$\epsilon E_1^{(0)} - (\dot{\epsilon} - \epsilon) E_2^{(0)} = -\frac{d}{dt} (\epsilon E_2^{(0)}) .$$

This is of the form

$$\sum_i E_i^{(0)} \delta x_i = \frac{dF}{dt}$$

with

$$\delta x_1 = \epsilon , \quad \delta x_2 = \epsilon - \dot{\epsilon} ,$$

which agrees with (2.5). This variation leaves the action invariant if $\epsilon(t_i) = 0$ ($i = 1, 2$).

Example 2

Consider the Lagrangian (2.6). The corresponding Euler derivative has the form (2.12), with

$$W^{(0)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} ,$$

and

$$\vec{K}^{(0)} = \begin{pmatrix} \dot{q}_2 e^{q_1} - e^{2q_1} \\ \dot{q}_3 - q_2 - \dot{q}_1 e^{q_1} \\ -\dot{q}_2 \end{pmatrix} .$$

$W^{(0)}$ possesses one left null-eigenvector $u^{(0)} = (1, 0, 0)$, implying a level zero constraint:

$$\phi^{(0)} \equiv \vec{u}^{(0)} \cdot \vec{E}^{(0)} = e^{q_1} (\dot{q}_2 - e^{q_1}) = 0 \quad (\text{on shell}) . \quad (2.28)$$

Next we search for further functions of the coordinates and velocities which vanish on shell, i.e. for $\vec{E}^{(0)} = 0$. To this effect we construct $\vec{E}^{(1)}$ as given by (2.15):

$$\vec{E}^{(1)} = \begin{pmatrix} \vec{E}^{(0)} \\ \frac{d}{dt} \phi^{(0)} \end{pmatrix} . \quad (2.29)$$

One then finds that $W^{(1)}$ and $\vec{K}^{(1)}$ in (2.16) are given by

$$W^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & e^{q_1} & 0 \end{pmatrix},$$

and

$$\vec{K}^{(1)} = \begin{pmatrix} \dot{q}_2 e^{q_1} - e^{2q_1} \\ \dot{q}_3 - q_2 - \dot{q}_1 e^{q_1} \\ -\dot{q}_2 \\ e^{q_1} \dot{q}_1 \dot{q}_2 - 2\dot{q}_1 e^{2q_1} \end{pmatrix}.$$

A new left-null eigenvector of $W^{(1)}$ at level one is clearly

$$\vec{u}^{(1)} = (0, -e^{q_1}, 0, 1),$$

and leads to a new constraint

$$\vec{u}^{(1)} \cdot \vec{E}^{(1)} = \dot{q}_1 \phi^{(0)} + e^{q_1} (q_2 - \dot{q}_3) = 0. \quad (2.30)$$

Since $\phi^{(0)} = 0$, we may choose instead for the new constraint function

$$\begin{aligned} \phi^{(1)} &= e^{q_1} (q_2 - \dot{q}_3) = \vec{u}^{(1)} \cdot \vec{E}^{(1)} - \dot{q}_1 \vec{u}^{(0)} \cdot \vec{E}^{(0)} \\ &= -e^{q_1} E_2^{(0)} + \frac{d}{dt} E_1^{(0)} - \dot{q}_1 E_1^{(0)}, \end{aligned} \quad (2.31)$$

which also vanishes on the space of solutions. Taking its time derivative, and constructing $\vec{E}^{(2)}$, $W^{(2)}$ and $\vec{K}^{(2)}$ one obtains

$$\vec{E}^{(2)} = \begin{pmatrix} \vec{E}^{(1)} \\ \frac{d}{dt} \phi^{(1)} \end{pmatrix}, \quad (2.32)$$

$$W^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & e^{q_1} & 0 \\ 0 & 0 & -e^{q_1} \end{pmatrix},$$

and

$$\vec{K}^{(2)} = \begin{pmatrix} \vec{K}^{(1)} \\ e^{q_1} (\dot{q}_1 q_2 - \dot{q}_1 q_3 + \dot{q}_2) \end{pmatrix}.$$

The matrix $W^{(2)}$ possesses a new left-zero mode of type v given by $\vec{v}^{(2)} = (0, 0, e^{q_1}, 0, 1)$. A simple calculation shows that this vector generates a gauge identity:

$$\vec{v}^{(2)} \cdot \vec{E}^{(2)} - \dot{q}_1 (\vec{u}^{(1)} \cdot \vec{E}^{(1)}) + \dot{q}_1^2 (\vec{u}^{(0)} \cdot \vec{E}^{(0)}) \equiv 0. \quad (2.33)$$

The final step now consists in reducing this expression to a form involving only the Euler derivative $\vec{E}^{(0)}$ and time derivatives thereof. Recalling (2.30) and (2.28), the gauge identity (2.33) can be rewritten as follows

$$e^{q_1} E_3^{(0)} - \frac{d}{dt} \left(e^{q_1} E_2^{(0)} + \dot{q}_1 E_1^{(0)} - \frac{d}{dt} E_1^{(0)} \right) + \dot{q}_1 \left(e^{q_1} E_2^{(0)} - \frac{d}{dt} E_1^{(0)} \right) + \dot{q}_1^2 E_1^{(0)} \equiv 0$$

Multiplying this identity with $\epsilon(t)$ one finds that it can be written in the form (2.23), with

$$F = \left[-e^{q_1} E_2^{(0)} - 2\dot{q}_1 E_1^{(0)} + \frac{d}{dt} E_1^{(0)} \right] \epsilon ,$$

and

$$\begin{aligned} \delta q_3 &= e^{q_1} \epsilon , \\ \delta q_2 &= e^{q_1} (\dot{\epsilon} + \epsilon \dot{q}_1) , \\ \delta q_1 &= \ddot{\epsilon} + 2\dot{q}_1 \dot{\epsilon} + \epsilon \ddot{q}_1 + \epsilon \dot{q}_1^2 . \end{aligned}$$

By setting $\epsilon = e^{-q_1} \alpha(t)$ we retrieve the transformation (2.7). The same would have been obtained, if we had replaced the constraint (2.28) by $\tilde{\phi}^{(1)} = \dot{q}_2 - e^{q_1}$, and allowed ϵ to be an explicit function of time only. This shows that we must allow the parameters to depend on time explicitly, as well as implicitly through the coordinates and their higher derivatives in time. As we have seen in our example, this is tied to the way we choose to define the constraints. Such redefinitions evidently presume some regularity conditions, ensuring that such reparametrizations will not imply the introduction of new constraints. We will come back to this point in chapter 6.

Example 3

We now consider an example of a system leading to two gauge identities. The associated symmetry transformations will hence involve two arbitrary functions.²

Consider the Lagrangian

$$L = \frac{1}{2} \dot{x}_1^2 + \dot{x}_1 (x_2 - x_3) + \frac{1}{2} (x_1 - x_2 + x_3)^2 .$$

For the Euler derivative (2.12) one finds that

$$W^{(0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ,$$

²As we will see in chapter 3 this corresponds to the number of “primary” constraints in the Hamiltonian formalism.

and

$$\vec{K}^{(0)} = \begin{pmatrix} \dot{x}_2 - \dot{x}_3 - x_1 + x_2 - x_3 \\ x_1 - x_2 + x_3 - \dot{x}_1 \\ x_2 - x_1 - x_3 + \dot{x}_1 \end{pmatrix} .$$

The matrix $W^{(0)}$ possesses two obvious (zero level) zero modes, which we now label as described in section 2:

$$\begin{aligned} \vec{u}^{(0,1)} &= (0, 1, 0) \\ \vec{u}^{(0,2)} &= (0, 0, 1) . \end{aligned}$$

Scalar product multiplication of the zero modes with the Euler derivative yields

$$\begin{aligned} \vec{u}^{(0,1)} \cdot \vec{E}^{(0)} &\equiv \phi^{(0,1)} , \\ \vec{u}^{(0,2)} \cdot \vec{E}^{(0)} &\equiv -\phi^{(0,1)} , \end{aligned}$$

where

$$\phi^{(0,1)} = x_1 - x_2 + x_3 - \dot{x}_1 .$$

Hence only one new constraint is generated, and we have one gauge identity,

$$\vec{u}^{(0,1)} \cdot \vec{E}^{(0)} + \vec{u}^{(0,2)} \cdot \vec{E}^{(0)} \equiv 0 ,$$

or

$$E_2^{(0)} + E_3^{(0)} \equiv 0 . \quad (2.34)$$

Proceeding with the Lagrangian algorithm we now construct $E^{(1)}$, $W^{(1)}$ and $\vec{K}^{(1)}$:

$$\begin{aligned} \vec{E}^{(1)} &= \begin{pmatrix} \vec{E}^{(0)} \\ \frac{d}{dt}\phi^{(0,1)} \end{pmatrix} , \\ W^{(1)} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} , \end{aligned}$$

and

$$\vec{K}^{(1)} = \begin{pmatrix} \vec{K}^{(0)} \\ \dot{x}_1 - \dot{x}_2 + \dot{x}_3 \end{pmatrix} .$$

$W^{(1)}$ leads to a new zero mode (in addition to those of $W^{(0)}$, augmented by a zero),

$$\vec{u}^{(1,1)} = (1, 0, 0, 1)$$

and to the constraint

$$\begin{aligned}\phi^{(1,1)} &= \vec{u}^{(1,1)} \cdot \vec{E}^{(1)} = E_1^{(1)} + E_4^{(1)} \\ &= \dot{x}_1 - x_1 + x_2 - x_3 = 0.\end{aligned}$$

We now notice that this is actually no new constraint. In fact

$$\phi^{(1,1)} + \phi^{(0,1)} \equiv 0.$$

This leads to a new gauge identity,

$$\vec{u}^{(1,1)} \cdot \vec{E}^{(1)} + \vec{u}_1^{(0,1)} \cdot \vec{E}^{(0)} = E_1^{(1)} + E_4^{(1)} + E_2^{(0)} \equiv 0. \quad (2.35)$$

Multiplying (2.34) and (2.35) by arbitrary functions $\alpha(t)$ and $\beta(t)$, respectively, and noting that $E_1^{(1)} = E_1^{(0)}$ and $E_4^{(1)} = \frac{d}{dt}E_2^{(0)}$ one is then led to the identity

$$\beta E_1^{(0)} + (\beta - \dot{\beta} + \alpha)E_2^{(0)} + \alpha E_3^{(0)} + \frac{d}{dt}(\beta E_2^{(0)}) \equiv 0.$$

Comparing this expression with (2.23), we conclude that the following transformations leave the action invariant:

$$\begin{aligned}\delta x_1 &= \beta, \\ \delta x_2 &= \beta - \dot{\beta} + \alpha, \\ \delta x_3 &= \alpha.\end{aligned} \quad (2.36)$$

2.5 Generator of gauge transformations and Noether identities

In the previous section we have related local symmetries of a Lagrangian to gauge (or Noether) identities. Our method allowed us to unravel systematically the local symmetries of a given action. In particular this method showed that in general time derivatives of gauge parameters are involved. The corresponding infinitesimal transformations (2.25) can also be written in the form

$$\delta q_i(t) = \int dt' R_a^i(t, t') \epsilon^{(a)}(t') \quad (2.37)$$

where $R_a^i(t, t')$ is a local kernel (generating function) whose explicit form can be read off from (2.25).

We now show that the operators

$$\Gamma_a(t') = -i \int dt R_a^i(t, t') \frac{\delta}{\delta q_i(t)}$$

can be regarded as the generators of the infinitesimal gauge transformation (2.25).³ For this it is convenient to compactify the notation by absorbing the time-dependences into the respective discrete indices ($(i, t) \rightarrow i$; $(a, t) \rightarrow a$), and omitting the integral-sign, which now becomes part of the summations. In this notation the variation of the action (2.8) becomes

$$\delta S = -E_i^{(0)} R_a^i \epsilon^a .$$

With “essential” parameters ϵ^a it follows that for a symmetry transformation (i.e., for $\delta S = 0$) we have the Noether identities⁴

$$E_i^{(0)} R_a^i \equiv 0 . \quad (2.38)$$

Similarly, following this notation, the infinitesimal transformations (2.37) can be written in the form

$$\delta q_i = i\epsilon^a \Gamma_a q_i ,$$

where

$$\Gamma_a = -i R_a^j \frac{\delta}{\delta q_j} .$$

A finite gauge transformation is then given by

$$\vec{q} \rightarrow \vec{G}_{(\lambda)}(\vec{q}) = e^{i\lambda^a \Gamma_a} \vec{q} .$$

The commutator of two generators can now be easily computed,⁵

$$\begin{aligned} [\Gamma_a, \Gamma_b] &= -[R_a^i \frac{\delta}{\delta q_i}, R_b^j \frac{\delta}{\delta q_j}] \\ &= -R_a^i \left(\frac{\delta}{\delta q_i} R_b^j \right) \frac{\delta}{\delta q_j} + R_b^j \left(\frac{\delta}{\delta q_j} R_a^i \right) \frac{\delta}{\delta q_i} \\ &= r_{ab}^i \frac{\delta}{\delta q_i} , \end{aligned}$$

where

$$r_{ab}^i = \left(R_b^j \frac{\delta}{\delta q_j} R_a^i - R_a^j \frac{\delta}{\delta q_j} R_b^i \right) . \quad (2.39)$$

³We follow here [Gitman 1990].

⁴By “essential parameters” we mean, that the matrix R_a^i possesses no right zero modes. In this case the theory is called “irreducible”. This is the case for our systematic construction, where all gauge identities are linearly independent. If the above matrix possesses right zero modes, then the number of Noether identities is reduced. One then distinguishes between “first stage”, “second stage”, etc. reducible theories (see [Gomis 1995]). Here we will consider only the irreducible case.

⁵We made use of the functional-derivative notation in order to remind the reader that the labels i, a etc. also include a continuous time variable. Thus summations will also include time-integrals.

By differentiating the Noether identities (2.38) with respect to q_j we obtain further useful identities:

$$\frac{\delta^2 S}{\delta q_j \delta q_i} R_a^i + \frac{\delta S}{\delta q_i} \frac{\delta R_a^i}{\delta q_j} \equiv 0 . \quad (2.40)$$

From here we see that on the space of physical trajectories ($\frac{\delta S}{\delta q_i} = 0$), $\frac{\delta^2 S}{\delta q_i \delta q_j}$ is degenerate, with R_a^j the right zero modes.

From the identities (2.40) and the definition (2.39), it follows that

$$\frac{\delta S}{\delta q_i} r_{ab}^i = 0 ,$$

or

$$E_i^{(0)} r_{ab}^i = 0 .$$

Hence (2.39) is also a generating function and will be of the form

$$r_{ab}^i = f_{ab}^c R_c^i + T_{ab}^i , \quad (2.41)$$

where

$$T_{ab}^i = a_{ab}^{ij} E_j^{(0)} ,$$

with a_{ab}^{ij} an antisymmetric tensor in the upper indices. We thus arrive at the algebra

$$[\Gamma_a, \Gamma_b] = i f_{ab}^c \Gamma_c + T_{ab}^i \frac{\delta}{\delta q_i} .$$

The second term on the rhs of (2.41) has the form of a *trivial* gauge transformation since $E_i^{(0)} T_{ab}^i \equiv 0$ by construction. If this term is absent, then

$$[\Gamma_a, \Gamma_b] = i f_{ab}^c \Gamma_c , \quad (2.42)$$

and one speaks of a “closed algebra”. If the structure functions f_{ab}^c depend on the “fields” q_i , one speaks of a “quasi group”. In the case where $f_{ab}^c = \text{const.}$, then one has a Lie algebra, and the finite transformations form a Lie group.

Under finite gauge transformations $\vec{q} \rightarrow \vec{G}_{(\lambda)}(\vec{q})$, an arbitrary function of $\vec{q}$ transforms as

$$F(\vec{q}) \rightarrow F(\vec{G}_{(\lambda)}(\vec{q})) = e^{i\lambda^a \Gamma_a} F(\vec{q}) .$$

In particular, for $F = [\frac{\delta S}{\delta q_i}]_{\vec{q} \rightarrow \vec{G}_{(\lambda)}(\vec{q})}$ and λ infinitesimal, we have that

$$\left. \frac{\delta S}{\delta q_i} \right|_{\vec{q} \rightarrow \vec{G}_\epsilon(\vec{q})} \approx \left(1 + \epsilon^a R_a^j \frac{\delta}{\delta q_j} \right) \frac{\delta S}{\delta q_i} .$$

Making use of the gauge identity (2.40),

$$\left. \frac{\delta S}{\delta q_i} \right|_{\vec{q} \rightarrow \vec{G}_\epsilon(\vec{q})} \approx \left(\delta_i^j - \epsilon^a \frac{\delta R_a^j}{\delta q_i} \right) \frac{\delta S}{\delta q_j}.$$

Hence it follows that

$$\frac{\delta S}{\delta q_j} = 0 \implies \left. \frac{\delta S}{\delta q_j} \right|_{\vec{q} \rightarrow \vec{G}_\lambda(\vec{q})} = 0;$$

i.e., the gauge identity (2.40) implies that if q^i is a possible trajectory, then this will also be the case for $\vec{q} \rightarrow \vec{G}_{(\lambda)}(\vec{q})$, for any λ .

Chapter 3

Hamiltonian Approach. The Dirac Formalism

3.1 Introduction

The quantization of non-singular systems is in principle straightforward. The classical Hamilton equations of motion are first written in Poisson bracket form. The corresponding quantum version is then obtained by replacing the Poisson brackets, multiplied by $i\hbar$, by commutators of the corresponding operators. On the other hand, the quantization of singular systems is non-trivial. This chapter is devoted to the introduction of the classical Hamiltonian framework within which such systems can be dealt with. This - as we have already pointed out - is not just an academic exercise, since all gauge field theories fall into the class of singular systems. The following discussion is based on the work of Dirac [Dirac 1950, 1964]. For further literature we refer the reader to [Hansen 1976, Sundermeyer 1982, Gitman 1990, Henneaux/Teitelboim 1992].

3.2 Primary constraints

Consider a system with a finite number of degrees of freedom $\{q_i\}$, whose dynamics is governed by a Lagrangian $L(q, \dot{q}) \equiv L(\{q_i\}, \{\dot{q}_i\})$. To arrive at a phase space formulation of the equations of motion one must construct the corresponding Hamiltonian. To this effect one first introduces the momenta canonically conjugate to the “coordinates” q_j ,

$$p_j = \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}_j} . \quad (3.1)$$

For non-singular systems these equations allow one to express the velocities $\dot{q}_i$ in terms of the canonical variables:

$$\dot{q}_i = f_i(q, p) .$$

The canonical Hamiltonian is then obtained by performing a Legendre transformation ¹

$$H_c(q, p) = \sum_i p_i f_i(q, p) - L(q, f(q, p)) ,$$

where the velocities have been expressed in terms of the canonical momenta and coordinates. The Hamilton equations of motion are then first order in the time derivative and take the standard form.

For the velocities to be expressible in terms of the coordinates and momenta one must have that $\det W \neq 0$, where the matrix W is the ‘‘Hessian’’ (2.2). In this case, as we have seen in chapter 2, the accelerations $\ddot{q}_i$ are uniquely determined by $\{q_i\}$ and $\{\dot{q}_i\}$. On the other hand, if $\det W = 0$, then W is not invertible. In this case the Hamilton equations of motion do not take the standard form and one speaks of a singular Lagrangian.

For the time being we shall assume that we are dealing with a finite number of degrees of freedom. The results can then, at least formally, be easily generalized to systems with a non-denumerable number of degrees of freedom.

Consider the Lagrangian

$$L = \frac{1}{2} \sum_{i,j=1}^n W_{ij}(q) \dot{q}_i \dot{q}_j + \sum_i \eta_i(q) \dot{q}_i - V(q) . \quad (3.2)$$

Since W_{ij} is a symmetric matrix, the canonical momenta are given by

$$p_i = \sum_j W_{ij}(q) \dot{q}_j + \eta_i(q) . \quad (3.3)$$

If W is an $n \times n$ singular matrix (i.e. $\det W = 0$) of rank R_W , then it possesses $n - R_W$ eigenvectors with vanishing eigenvalue. Label these eigenvectors by $\vec{v}^{(\alpha)}$. Then

$$\sum_j W_{ij}(q) v_j^{(\alpha)}(q) = 0 \quad , \quad \alpha = 1, \dots, n - R_W .$$

It then follows from (3.3) that there exist $n - R_W$ constraints

$$\sum_i v_i^{(\alpha)}(q) (p_i - \eta_i(q)) = 0 ,$$

¹In this chapter we use throughout lower indices for convenience, at the expense of introducing explicitly summation signs.

which follow solely from the definition of the canonical momenta (3.1). Let $\{p_a\}$ denote a set of R_W linearly independent momenta. The remaining set of $n - R_W$ momenta we denote with a Greek subscript: $\{p_\alpha\}$. Then the constraint equations will be of the form

$$\sum_{\beta} M_{\alpha\beta}(q)p_{\beta} - F_{\alpha}(q, \{p_a\}) = 0 \quad (3.4)$$

with

$$M_{\alpha\beta}(q) = v_{\beta}^{(\alpha)}(q),$$

and

$$F_{\alpha}(q, \{p_b\}) = \sum_i v_i^{(\alpha)}(q)\eta_i(q) - \sum_b v_b^{(\alpha)}(q)p_b.$$

M is necessarily an invertible matrix since, if the determinant of M would vanish, then M would possess eigenvectors with vanishing eigenvalues, implying the existence of further constraints. Hence the number of constraints would be larger than $n - R_W$.

Since $\det M \neq 0$, the constraints (3.4) can also be written in the *standard* form

$$\phi_{\alpha}(q, p) := p_{\alpha} - g_{\alpha}(q, \{p_a\}) = 0, \quad (3.5)$$

where

$$g_{\alpha} = \sum_{\beta} M_{\alpha\beta}^{-1} F_{\beta}.$$

Written in this form, the label α on the constraint ϕ_{α} is that of the momentum p_{α} , conjugate to the velocity $\dot{q}_{\alpha}$. The constraints (3.5) are referred to in Dirac's terminology as *primary constraints*. They follow solely from the definition of the canonical momenta, without making use of the equations of motion. The primary constraints have no analog on the Lagrangian level. On the Hamiltonian level they define a $2n - (n - R_W) = n + R_W$ -dimensional subspace Γ_P of the $2n$ -dimensional phase space.

Although the above considerations have been based on a Lagrangian quadratic in the velocities, (3.5) is generally valid for Lagrangians which depend only on q and $\dot{q}$, and no higher derivatives of q (the Hessian (2.2) can thus in principle also depend on the velocities). This can be seen as follows:

Let W_{ab} , ($a, b = 1, \dots, R_W$) be the largest invertible submatrix of W_{ij} , where a suitable rearrangement of the components has been carried out. We can then solve eqs. (3.1) for R_W velocities $\dot{q}_a$ in terms of the coordinates q_i , the momenta $\{p_a\}$ and the remaining velocities $\{\dot{q}_{\alpha}\}$:

$$\dot{q}_a = f_a(q, \{p_b\}, \{\dot{q}_{\beta}\}), \quad a, b = 1, \dots, R_W; \quad \beta = R_W + 1, \dots, n. \quad (3.6)$$

Inserting this expression into (3.1), one arrives at a relation of the form,

$$p_j = h_j(q, \{p_a\}, \{\dot{q}_\alpha\}) .$$

For $j = a$ ($a = 1, \dots, R_W$) this relation must necessarily reduce to an identity. The remaining equations read

$$p_\alpha = h_\alpha(q, \{p_a\}, \{\dot{q}_\beta\}) .$$

But the rhs cannot depend on the velocities $\dot{q}_\beta$, since otherwise we could express still more velocities from the set $\{\dot{q}_\alpha\}$ in terms of the coordinates, the momenta and the remaining velocities, which is not possible.

We now embark on the construction of the Hamilton equations of motion for such a constraint system, following first a conventional approach, and thereafter another line of reasoning, which emphasizes the role played by the zero modes of W .

3.3 The Hamilton equations of motion

We first prove the following propositions [Sudarshan 1974]:

Proposition 1

On the subspace Γ_P defined by (3.5), the canonical Hamiltonian is only a function of the coordinates $\{q_i\}$ and the momenta $\{p_a\}$; i.e., it does not depend on the velocities $\{\dot{q}_\alpha\}$.

Proof

Consider the canonical Hamiltonian on the subspace Γ_P :

$$H_0 \equiv H_c|_{\Gamma_P} = \sum_a p_a f_a + \sum_a g_a \dot{q}_a - L(q, \{f_b\}, \{\dot{q}_\beta\}) , \quad (3.7)$$

where we have used (3.5) and (3.6). The rhs of (3.7) defines a function which in principle depends on $\{q_i\}$, $\{p_a\}$ and the velocities $\{\dot{q}_\alpha\}$. We wish to show that H_0 does not depend on the velocities $\{\dot{q}_\alpha\}$. Indeed

$$\begin{aligned} \frac{\partial H_0}{\partial \dot{q}_\beta} &= \sum_a p_a \frac{\partial f_a}{\partial \dot{q}_\beta} + g_\beta - \left(\frac{\partial L}{\partial \dot{q}_a} \right)_{\dot{q}_a=f_a} \frac{\partial f_a}{\partial \dot{q}_\beta} - \left(\frac{\partial L}{\partial \dot{q}_\beta} \right)_{\dot{q}_a=f_a} \\ &= \sum_a \left(p_a - \left(\frac{\partial L}{\partial \dot{q}_a} \right)_{\dot{q}_a=f_a} \right) \frac{\partial f_a}{\partial \dot{q}_\beta} + \left(g_\beta - \left(\frac{\partial L}{\partial \dot{q}_\beta} \right)_{\dot{q}_a=f_a} \right) = 0 . \end{aligned}$$

Hence it follows that

$$H_0 = H_0(q, \{p_a\}). \quad (3.8)$$

Note that we have not used the Lagrange equations of motion in the proof.

Proposition 2

In the presence of primary constraints (3.5), the Hamilton equations of motion read:

$$\begin{aligned} \dot{q}_i &= \frac{\partial H_0}{\partial p_i} + \sum_{\beta} \dot{q}_{\beta} \frac{\partial \phi_{\beta}}{\partial p_i}, \\ \dot{p}_i &= -\frac{\partial H_0}{\partial q_i} - \sum_{\beta} \dot{q}_{\beta} \frac{\partial \phi_{\beta}}{\partial q_i}, \\ \phi_{\alpha}(q, p) &= 0, \end{aligned} \quad (3.9)$$

where the $\dot{q}_{\beta}$ are (a priori) undetermined velocities. Note that we are only allowed to set $\phi_{\alpha} = 0$ after having performed the partial differentiations.

Proof

From (3.7) we obtain, using the fact that H_0 does not depend on $\dot{q}_{\alpha}$,

$$\frac{\partial H_0}{\partial p_a} = f_a + \sum_b p_b \frac{\partial f_b}{\partial p_a} + \sum_{\beta} \frac{\partial g_{\beta}}{\partial p_a} \dot{q}_{\beta} - \left(\frac{\partial L}{\partial \dot{q}_b} \right)_{\dot{q}_a=f_a} \frac{\partial f_b}{\partial p_a},$$

or

$$\frac{\partial H_0}{\partial p_a} = \dot{q}_a + \sum_{\beta} \frac{\partial g_{\beta}}{\partial p_a} \dot{q}_{\beta}. \quad (3.10)$$

Furthermore

$$\begin{aligned} \frac{\partial H_0}{\partial q_i} &= \sum_b p_b \frac{\partial f_b}{\partial q_i} + \sum_{\beta} \frac{\partial g_{\beta}}{\partial q_i} \dot{q}_{\beta} - \left(\frac{\partial L}{\partial q_i} \right)_{\dot{q}_a=f_a} - \sum_b \left(\frac{\partial L}{\partial \dot{q}_b} \right)_{\dot{q}_b=f_b} \frac{\partial f_b}{\partial q_i} \\ &= - \left(\frac{\partial L}{\partial q_i} \right)_{\{\dot{q}_a=f_a\}} + \sum_{\beta} \frac{\partial g_{\beta}}{\partial q_i} \dot{q}_{\beta} \\ &= - \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right)_{\{\dot{q}_a=f_a\}} + \sum_{\beta} \frac{\partial g_{\beta}}{\partial q_i} \dot{q}_{\beta}, \end{aligned}$$

or

$$\frac{\partial H_0}{\partial q_i} = -\dot{p}_i + \sum_i \frac{\partial g_{\beta}}{\partial q_i} \dot{q}_{\beta}. \quad (3.11)$$

Note that this time we have made use of the Euler-Lagrange equations of motion.

Equations (3.10) and (3.11) are asymmetric in the number of coordinates and momenta, and their time derivatives. A more symmetric form is obtained by noting that H_0 does not depend on $\{p_\alpha\}$. Making use of the definition of ϕ_α (cf. eq. (3.5)) we first rewrite the above equations as follows:

$$\dot{q}_a = \frac{\partial H_0}{\partial p_a} + \sum_{\beta} \dot{q}_{\beta} \frac{\partial \phi_{\beta}}{\partial p_a}, \quad (3.12)$$

$$\dot{p}_i = -\frac{\partial H_0}{\partial q_i} - \sum_{\beta} \dot{q}_{\beta} \frac{\partial \phi_{\beta}}{\partial q_i}. \quad (3.13)$$

Because $\frac{\partial H_0}{\partial p_\alpha} = 0$, and $\frac{\partial \phi_{\beta}}{\partial p_\alpha} = \delta_{\beta\alpha}$, we can supplement these equations by the identities

$$\dot{q}_\alpha \equiv \frac{\partial H_0}{\partial p_\alpha} + \sum_{\beta} \dot{q}_{\beta} \frac{\partial \phi_{\beta}}{\partial p_\alpha}.$$

Adjoining these equations to (3.12), we arrive at (3.9).

It is not obvious that the equations for $\dot{q}_i$ and $\dot{p}_i$ are consistent with the primary constraints (3.5). Thus, for consistency we must also have

$$\dot{p}_\alpha = \frac{d}{dt} g_\alpha(q, \{p_a\}),$$

where $\dot{p}_\alpha$ is given by the rhs of (3.13), together with the primary constraints. This may lead to so-called “secondary constraints”. These constraints must in turn be consistent with the equations of motion and primary constraints (and so forth). We shall soon look at this problem in more detail. In fact, as we will see, the full set of Lagrange equations of motion will only become manifest on Hamiltonian level once secondary constraints are taken into account. This does not mean, however, that they must be added explicitly to the Hamilton equations of motion (3.9), since they are hidden in the primary constraints, which are required to hold for all times.

3.3.1 Streamlining the Hamilton equations of motion

For the following discussion it is useful to introduce the concept of a weak equality as defined by Dirac:

Definition: f is weakly equal to g ($f \approx g$) if f and g are equal on the subspace defined by the primary constraints $\phi_\beta = 0$: $f|_{\Gamma_P} = g|_{\Gamma_P}$.

In the following $f(q, p) \approx g(q, p)$ will thus be a short hand for the following set of equations,

$$f(q, p) = g(q, p) , \quad \phi_\alpha(q, p) = 0 .$$

With this definition we can write the Hamilton equations (3.9) in the form

$$\dot{q}_i \approx \frac{\partial H_T}{\partial p_i} , \quad \dot{p}_i \approx -\frac{\partial H_T}{\partial q_i} , \quad (3.14)$$

where the “total Hamiltonian” H_T is defined by

$$H_T = H_0 + \sum_{\alpha} v_{\alpha} \phi_{\alpha} . \quad (3.15)$$

Note that the derivatives in (3.14) do not act on the v_{α} since the expressions are to be evaluated on the primary constrained surface, as is implied by the “weak equality” sign. In fact, since the “velocities” v_{α} are a priori not given functions of the canonical variables, the action of the derivatives would not even be defined.

Making use of the definition of “weak equality”, we can further relax the definition of H_T . To this end we first prove the following proposition: [Sudarshan 1974]:

Proposition 3

If f and h are two functions defined over the entire phase space Γ , spanned by $\{q_i\}$ and $\{p_i\}$, and if $f(q, p)|_{\Gamma_P} = h(q, p)|_{\Gamma_P}$, then

$$\begin{aligned} \frac{\partial}{\partial q_i} \left(f - \sum_{\beta} \phi_{\beta} \frac{\partial f}{\partial p_{\beta}} \right) &\approx \frac{\partial}{\partial q_i} \left(h - \sum_{\beta} \phi_{\beta} \frac{\partial h}{\partial p_{\beta}} \right) \\ \frac{\partial}{\partial p_i} \left(f - \sum_{\beta} \phi_{\beta} \frac{\partial f}{\partial p_{\beta}} \right) &\approx \frac{\partial}{\partial p_i} \left(h - \sum_{\beta} \phi_{\beta} \frac{\partial h}{\partial p_{\beta}} \right) \end{aligned} \quad (3.16)$$

Proof:

Consider two functions $f(q, \{p_a\}, \{p_{\alpha}\})$ and $h(q, \{p_a\}, \{p_{\alpha}\})$. By assumption,

$$f(q, \{p_a\}, \{g_{\alpha}\}) = h(q, \{p_a\}, \{g_{\alpha}\})$$

where we made use of (3.5). From here it follows that

$$\left(\frac{\partial f}{\partial q_i} + \sum_{\beta} \frac{\partial f}{\partial p_{\beta}} \frac{\partial g_{\beta}}{\partial q_i} \right)_{\Gamma_P} = \left(\frac{\partial h}{\partial q_i} + \sum_{\beta} \frac{\partial h}{\partial p_{\beta}} \frac{\partial g_{\beta}}{\partial q_i} \right)_{\Gamma_P}$$



and

$$\left(\frac{\partial f}{\partial p_a} + \sum_{\beta} \frac{\partial f}{\partial p_{\beta}} \frac{\partial g_{\beta}}{\partial p_a} \right)_{\Gamma_P} = \left(\frac{\partial h}{\partial p_a} + \sum_{\beta} \frac{\partial h}{\partial p_{\beta}} \frac{\partial g_{\beta}}{\partial p_a} \right)_{\Gamma_P}.$$

Making use of (3.5), and of the definition of the weak equality, these equations can also be written in the form

$$\begin{aligned} \frac{\partial}{\partial q_i} \left(f - \sum_{\beta} \phi_{\beta} \frac{\partial f}{\partial p_{\beta}} \right) &\approx \frac{\partial}{\partial q_i} \left(h - \sum_{\beta} \phi_{\beta} \frac{\partial h}{\partial p_{\beta}} \right), \\ \frac{\partial}{\partial p_a} \left(f - \sum_{\beta} \phi_{\beta} \frac{\partial f}{\partial p_{\beta}} \right) &\approx \frac{\partial}{\partial p_a} \left(h - \sum_{\beta} \phi_{\beta} \frac{\partial h}{\partial p_{\beta}} \right). \end{aligned}$$

Finally, since $\frac{\partial \phi_{\beta}}{\partial p_{\alpha}} = \delta_{\beta\alpha}$ we also trivially have that

$$\frac{\partial}{\partial p_{\alpha}} \left(f - \sum_{\beta} \phi_{\beta} \frac{\partial f}{\partial p_{\beta}} \right) \approx \frac{\partial}{\partial p_{\alpha}} \left(h - \sum_{\beta} \phi_{\beta} \frac{\partial h}{\partial p_{\beta}} \right).$$

Hence we are led to (3.16).

Corollary:

From (3.16) it follows in particular that

$$\begin{aligned} \dot{q}_i &\approx \frac{\partial H}{\partial p_i} + \sum_{\beta} v^{\beta} \frac{\partial \phi_{\beta}}{\partial p_i}, \\ \dot{p}_i &\approx -\frac{\partial H}{\partial q_i} - \sum_{\beta} v^{\beta} \frac{\partial \phi_{\beta}}{\partial q_i}, \end{aligned} \quad (3.17)$$

where $H \approx H_0$, and v^{β} are undetermined parameters.

Proof:

Consider two functions $H(\{q_i\}, \{p_i\})$ and $H_0(\{q_i\}, \{p_a\})$, satisfying

$$H(\{q_i\}, \{p_i\}) \approx H_0(\{q_i\}, \{p_a\}).$$

Setting $f = H_0$ and $h = H$ in (3.16), it follows from (3.8) that

$$\begin{aligned} \frac{\partial H_0}{\partial q_i} &\approx \frac{\partial}{\partial q_i} \left(H - \sum_{\rho} \phi_{\rho} \frac{\partial H}{\partial p_{\rho}} \right), \\ \frac{\partial H_0}{\partial p_i} &\approx \frac{\partial}{\partial p_i} \left(H - \sum_{\rho} \phi_{\rho} \frac{\partial H}{\partial p_{\rho}} \right). \end{aligned}$$

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Inserting these expressions in (3.9), we obtain

$$\begin{aligned}\dot{q}_i &\approx \frac{\partial}{\partial p_i} \left(H - \sum_{\beta} \phi_{\beta} \frac{\partial H}{\partial p_{\beta}} \right) + \sum_{\beta} \dot{q}_{\beta} \frac{\partial \phi_{\beta}}{\partial p_i} , \\ \dot{p}_i &\approx -\frac{\partial}{\partial q_i} \left(H - \sum_{\beta} \phi_{\beta} \frac{\partial H}{\partial p_{\beta}} \right) - \sum_{\beta} \dot{q}_{\beta} \frac{\partial \phi_{\beta}}{\partial q_i} .\end{aligned}$$

These expressions can also be written in the form ²

$$\begin{aligned}\dot{q}_i &\approx \frac{\partial}{\partial p_i} \left[H + \sum_{\beta} \left(\dot{q}_{\beta} - \frac{\partial H}{\partial p_{\beta}} \right) \phi_{\beta} \right] , \\ \dot{p}_i &\approx -\frac{\partial}{\partial q_i} \left[H + \sum_{\beta} \left(\dot{q}_{\beta} - \frac{\partial H}{\partial p_{\beta}} \right) \phi_{\beta} \right] .\end{aligned}$$

Defining

$$v_{\beta} := \dot{q}_{\beta} - \frac{\partial H}{\partial p_{\beta}} , \quad (3.18)$$

equations (3.17) follow. These equations can also be written in the form

$$\dot{q}_i \approx \frac{\partial H_T}{\partial p_i} , \quad \dot{p}_i \approx -\frac{\partial H_T}{\partial q_i} , \quad (3.19)$$

where H_T is the *total* Hamiltonian (3.15) with H_0 replaced by H :

$$H_T = H + \sum_{\beta} v_{\beta} \phi_{\beta} , \quad H \approx H_0 . \quad (3.20)$$

Hence the Hamilton equations of motion take a form analogous to those for an unconstrained system. The canonical Hamiltonian is merely replaced by the total Hamiltonian (3.20), where $H(q, p)$ is now any function of the canonical variables which on the primary constrained surface Γ_P reduces to the canonical Hamiltonian evaluated on this surface. Note that in the case where $H = H_0$ we have from (3.18) and (3.8) that $v_{\beta} = \dot{q}_{\beta}$.

Equations (3.19) can also be written in Poisson bracket form. Defining the Poisson bracket of $A(q, p)$ and $B(q, p)$ in the usual way, ³

$$\{A, B\} = \sum_i \left(\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right) \quad (3.21)$$

²Note again that the action of the partial derivatives on the $\dot{q}_{\alpha}$'s is not defined. Such terms are however proportional to the constraints and therefore vanish on the primary surface.

³In this chapter we restrict ourselves to commuting c-number variables.

we have that

$$\dot{q}_i \approx \{q_i, H_T\}, \quad \dot{p}_i \approx \{p_i, H_T\}. \quad (3.22)$$

For later purposes we recall some fundamental properties of Poisson brackets:

i) *Antisymmetry*

$$\{f, g\} = -\{g, f\}$$

ii) *Linearity*

$$\{f_1 + f_2, g\} = \{f_1, g\} + \{f_2, g\}$$

iii) *Associative product law*

$$\{f_1 f_2, g\} = f_1 \{f_2, g\} + \{f_1, g\} f_2$$

iv) *Jacobi-identity (bosonic)*

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0. \quad (3.23)$$

From these properties it follows that for any function of the phase space variables $F(q, p)$,

$$\dot{F} \approx \{F, H_T\}.$$

3.3.2 Alternative derivation of the Hamilton equations

Before proceeding with our Hamiltonian analysis of singular systems, it is instructive to rederive the above equations of motion by starting from a non-singular Lagrangian depending on one or more parameters, for which the Hamilton equations of motion are the standard ones, and which reduces to the singular Lagrangian of interest by taking an appropriate limit [Rothe 2003a]. This, as we shall see, will also shed light on the role played by the primary constraints.

Consider the following Lagrangian,

$$L = \frac{1}{2} \sum_{i,j} W_{ij}(q; \alpha) \dot{q}_i \dot{q}_j - \sum_i \eta_i(q) \dot{q}_i - V(q), \quad (3.24)$$

where q stands for the set of coordinates $\{q_i\}$, and α stands collectively for a set of parameters. For $\alpha \neq \alpha_c$ we assume that $\det W \neq 0$, so that we are dealing with a non-singular system. We assume that the singular system of interest is realized for $\alpha = \alpha_c$ (the subscript c stands for “critical”), where $\det W(q; \alpha_c) = 0$.⁴ Our aim is to derive the Hamilton equations of motion

⁴Clearly the form of (3.24) is not unique.

for the singular system, by taking the limit $\alpha \rightarrow \alpha_c$ of the Hamilton equations following from (3.24).

For $\alpha \neq \alpha_c$, one has for the Legendre transform of (3.24) the canonical Hamiltonian,

$$H_c = \frac{1}{2} \sum_{i,j} (p_i + \eta_i) W_{ij}^{-1} (p_j + \eta_j) + V(q) , \quad (3.25)$$

where the canonical momenta are related to the velocities by

$$p_i = \sum_j W_{ij}(q; \alpha) \dot{q}_j - \eta_i(q) . \quad (3.26)$$

The symmetric matrix W can be diagonalized by an orthogonal transformation $W \rightarrow W_D = C^T W C$. In terms of the eigenvalues and orthonormalized eigenvectors, W can be written in the form

$$W_{ij} = \sum_{\ell} \lambda_{\ell} v_i^{(\ell)} v_j^{(\ell)} ,$$

where

$$W(q; \alpha) \vec{v}^{(\ell)}(q; \alpha) = \lambda_{\ell}(q; \alpha) \vec{v}^{(\ell)}(q; \alpha) , \quad (3.27)$$

and where by assumption, $\lambda_{\ell}(q; \alpha) \neq 0$ for $\alpha \neq \alpha_c$. Correspondingly W_{ij}^{-1} is obtained by making the replacement $\lambda_{\ell} \rightarrow \frac{1}{\lambda_{\ell}}$. The Hamiltonian (3.25) is thus given by

$$H_c = \frac{1}{2} \sum_{\ell} \frac{1}{\lambda_{\ell}} \phi_{\ell}^2 + V(q) , \quad (3.28)$$

where

$$\phi_{\ell}(q, p; \alpha) := (\vec{p} + \vec{\eta}(q)) \cdot \vec{v}^{(\ell)}(q; \alpha) . \quad (3.29)$$

For $\alpha \neq \alpha_c$ the Hamilton equations of motion then take the form

$$\dot{q}_i = \frac{\partial H_c}{\partial p_i} = \sum_{\ell} \frac{1}{\lambda_{\ell}} \phi_{\ell} \frac{\partial \phi_{\ell}}{\partial p_i} , \quad (3.30)$$

$$\dot{p}_i = -\frac{\partial H_c}{\partial q_i} = -\frac{1}{2} \sum_{\ell} \frac{\partial}{\partial q_i} \left(\frac{1}{\lambda_{\ell}} \phi_{\ell}^2 \right) - \frac{\partial V}{\partial q_i} . \quad (3.31)$$

Consider now the limit $\alpha \rightarrow \alpha_c$ where $\det W \rightarrow 0$. Let $\{\lambda_{\ell_0}\}$ denote the set of eigenvalues which vanish in this limit. In order to implement the limit we first write (3.30) and (3.31) in the form

$$\dot{q}_i = \sum_{\ell \neq \{\ell_0\}} \frac{1}{\lambda_{\ell}} \phi_{\ell} \frac{\partial \phi_{\ell}}{\partial p_i} + \sum_{\ell_0} \frac{1}{\lambda_{\ell_0}} \phi_{\ell_0} \frac{\partial \phi_{\ell_0}}{\partial p_i} , \quad (3.32)$$

$$\dot{p}_i = -\frac{1}{2} \sum_{\ell \neq \{\ell_0\}} \frac{\partial}{\partial q_i} \left(\frac{1}{\lambda_\ell} \phi_\ell^2 \right) - \frac{1}{2} \sum_{\ell_0} \frac{\partial}{\partial q_i} \left(\frac{1}{\lambda_{\ell_0}} \phi_{\ell_0}^2 \right) - \frac{\partial V}{\partial q_i}. \quad (3.33)$$

From (3.29), (3.26) and (3.27) we have that

$$\phi_{\ell_0} = \sum_{i,j} v_i^{(\ell_0)} W_{ij} \dot{q}_j = \lambda_{\ell_0} (\vec{v}^{(\ell_0)} \cdot \dot{\vec{q}}). \quad (3.34)$$

Hence

$$\lim_{\alpha \rightarrow \alpha_c} \frac{1}{\lambda_{\ell_0}} \phi_{\ell_0} = \rho_{\ell_0}(q, \dot{q}; \alpha_c), \quad (3.35)$$

where

$$\rho_{\ell_0}(q, \dot{q}; \alpha_c) = \dot{\vec{q}} \cdot \vec{v}^{(\ell_0)}(q; \alpha_c). \quad (3.36)$$

Note that the finiteness of the velocities in (3.34) implies that

$$\phi_{\ell_0}(q, p, \alpha_c) = 0, \quad (3.37)$$

where ϕ_{ℓ_0} is given by (3.29) with $\ell = \ell_0$. These are just the primary constraints.

Hence for $\alpha \rightarrow \alpha_c$, equation (3.32) reduces to

$$\dot{q}_i = \sum_{\ell \neq \{\ell_0\}} \left(\frac{1}{\lambda_\ell} \phi_\ell \frac{\partial \phi_\ell}{\partial p_i} \right)_{\alpha=\alpha_c} + \sum_{\ell_0} \rho_{\ell_0} \left(\frac{\partial \phi_{\ell_0}}{\partial p_i} \right)_{\alpha=\alpha_c} \quad (3.38)$$

with ρ_{ℓ_0} given by (3.36). Consider next eqs. (3.33). Again making use of (3.35) and the fact that $\frac{\partial \lambda_{\ell_0}}{\partial q_i} |_{\alpha=\alpha_c} = 0$, these reduce for $\alpha \rightarrow \alpha_c$ to

$$\dot{p}_i = -\frac{1}{2} \sum_{\ell \neq \{\ell_0\}} \left[\frac{\partial}{\partial q_i} \left(\frac{1}{\lambda_\ell} \phi_\ell^2 \right) \right]_{\alpha=\alpha_c} - \sum_{\ell_0} \rho_{\ell_0} \left(\frac{\partial \phi_{\ell_0}}{\partial q_i} \right)_{\alpha=\alpha_c} - \frac{\partial V}{\partial q_i}, \quad (3.39)$$

where use has been made of (3.36).

We now notice that the first sum on the rhs of eqs. (3.38) and (3.39) are just given by $\frac{\partial H_0}{\partial p_i}$ and $-\frac{\partial H_0}{\partial q_i}$, where H_0 is the Hamiltonian obtained from (3.28) by taking the limit $\alpha \rightarrow \alpha_c$:

$$H_0 = \frac{1}{2} \sum_{\ell \neq \ell_0} \frac{1}{\lambda_\ell(q, \alpha_c)} \phi_\ell^2(q, p; \alpha_c) + V(q) \quad (3.40)$$

Here use has again been made of the fact that $\lim_{\alpha \rightarrow \alpha_c} \frac{\phi_{\ell_0}}{\lambda_{\ell_0}}$ is finite, as well as of (3.37). H_0 is just the canonical Hamiltonian derived from the (3.24) with $\alpha = \alpha_c$, evaluated on the surface (3.37). Hence, together with the constraint equations (3.37), the equations of motion (3.38) and (3.39) take the form

$$\dot{q}_i \approx \frac{\partial H_T}{\partial p_i}, \quad \dot{p}_i \approx -\frac{\partial H_T}{\partial q_i} \quad (3.41)$$