



World Scientific Series in Contemporary Chemical Physics – Vol. 22

Energy and Geometry

An Introduction to Deformed Special Relativity



Fabio Cardone
Roberto Mignani



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To our parents

Maria Pia, Mario Nietta^a, Pietro

^aAfter the completion of the book, the mother of R.M. passed in May 2003. R.M. dedicates this book to her memory.

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Preface

In 1955 the Italian mathematician Bruno Finzi, in his contribution to the book “*Fifty Years of Relativity*”,^b stated his “*Principle of Solidarity*” (PS),^c that sounds “*It’s (indeed) necessary to consider space-time TO BE SOLIDLY CONNECTED with the physical phenomena occurring in it, so that its features and its very nature do change with the features and the nature of those. In this way not only (as in classical and special-relativistic physics) space-time properties affect phenomena, but reciprocally phenomena do affect space-time properties. One thus recognizes in such an appealing ‘Principle of Solidarity’ between phenomena and space-time that characteristic of mutual dependence between entities, which is peculiar to modern science.*” Moreover, referring to a generical N-dimensional space: “*It can, a priori, be pseudoeuclidean, riemannian, non-riemannian. But — he wonders — how is indeed the space-time where physical phenomena take place? Pseudoeuclidean, riemannian, non-riemannian, according to their nature, as requested by the principle of solidarity between space-time and phenomena occurring in it.*”

Of course, Finzi’s main purpose was to apply such a principle to Einstein’s Theory of General Relativity, namely to the class of gravitational phenomena. However, its formulation is as general as possible, so to apply in principle to all the known physical interactions. Therefore, Finzi’s PS is at the very ground of any attempt at geometrizing physics, i.e. describing physical forces in terms of the geometrical structure of space-time.

^bB. Finzi: “Relatività Generale e Teorie Unitarie”, in “*Cinquant’anni di Relatività*”, ed. M. Pantaleo (Giunti, Firenze, Italy, 1955), p. 194.

^cIt is quite difficult to express in English in a simple way the Italian words “*solidarietà*” and “*solidale*”, used by Finzi to mean the feedback between space-time and interactions. A possible way to render them is to use “*solidarity*” and “*solidly connected*”, respectively — at the price of partially loosing the common root of the Italian words — with the warning that what Finzi really means is that the very structure of space-time is determined by the physical phenomena which do take place in it.

Such a project (pioneered by Einstein himself) revealed itself unsuccessful even when only two interactions were known, the electromagnetic and the gravitational one. It was fully abandoned starting from the middle of the 20th century, due to the discovery of the two nuclear interactions, the weak and the strong one (apart from recent attempts based on string theory).

Actually, the very euclidean geometry is not only the first historical example of self-consistent and complete, axiomatic system of mathematics, but does represent the first theory of physical reality (as stressed for instance by Penrose). It describes in a quantitative way, in mathematical language, the relations among measured physical entities — distances, in this case — and therefore of the physical space in which phenomena occur.

However, the measurement of distances depends on the motion of the body which actually performs the measurement. Such a dependence is indeed not on the *kind* of motion, but rather on the *energy* needed to let the body move, and on the *interaction* providing such energy. The measurement of time needs as well a periodic motion with constant frequency, and therefore it too depends on the energy and on the interaction.

This simple example shows how *energy does play a fundamental role in determining the very geometrical structure of space-time* (in analogy with the General-Relativistic case, where — as is well known — the energy-momentum tensor is the source of the gravitational field).

By starting from such considerations, a possible way to implement Finzi's principle for *all* fundamental interactions is provided by the formalism of *Deformed Special Relativity (DSR)* developed in the last decade. It is based on a *deformation* of the Minkowski space, namely a space-time endowed with a metric whose coefficients just depend on energy. Such an energy-dependent metric does assume a *dynamical role*, thus providing a geometrical description of the fundamental interaction considered and implementing the feedback between space-time structure and physical interactions which is just the content and the heritage of Finzi's principle.

Aim of this book is to acquaint the reader with the basic concepts and methods of DSR. Part I and II contain the physical and mathematical foundations of the theory. In Part III, we discuss the physical experiments whose analysis allows one to derive the explicit form of the phenomenological metrics for all four fundamental interactions, and some of the physical applications of the DSR formalism. The connection of DSR with the problem of the breakdown of local Lorentz invariance is considered in Part IV.

The natural link between DSR and a generalized, five-dimensional Kaluza-Klein scheme is also outlined.

We are indebted of course to many colleagues and friends. First of all, it is both a duty and a pleasure to express our special gratitude to Myron Evans, who was so kind to invite us to write this book for the series “*Advances in Chemical Physics*” of which he is the Editor in Chief. Part of the content of the book was the object of a one-month lecture course delivered by F.C. in spring 2002 at the Mathematical Department of Messina University, on invitation by Enzo Ciancio and Liliana Restuccia, and we are pleased to thank them warmly. We are very grateful to the co-authors of all our papers, in particular to Umberto Bartocci, Mauro Francaviglia, Mario Gaspero, Alessio Marrani, Vladislav S. Olkhovsky, Eliano Pessa. Thanks are also due to Carol O. Alley, Tito Arecchi, Gaetano Caricato, Sidney Coleman, Sheldon L. Glashow, Roman Jackiw, Maxim Yu. Khlopov, Rostislav V. Konoplich, Daniela Mugnai, Fabio Pistella and Anedio Ranfagni for kind interest and useful discussions. Needless to say, the responsibility for any errors or omission rests with the authors.

*Fabio Cardone
Roberto Mignani
Roma, May 2003*

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Contents

<i>Dedication</i>	v
<i>Preface</i>	vii
I Generalizing Special Relativity	1
1 An Axiomatic View to Special Relativity	3
1.1 Foundations of Special Relativity	3
1.2 The Principle of Solidarity	4
1.3 Locality and Interaction	5
1.3.1 Locality and Nonlocality	5
1.3.2 Potential and Nonpotential Interactions	6
2 Deformed Minkowski Space-Time	7
3 Description of Interactions by Energy-Dependent Metrics	11
3.1 Energy and the Finzi Principle	11
3.2 Energy as Dynamical Variable	13
II Relativity in a Deformed Space-Time	19
4 Generalized Principle of Relativity and Lorentz Transformations	21
4.1 Deformed Special Relativity	21
4.2 Deriving the Deformed Lorentz Transformations	21
4.3 Maximal Causal Speed Revisited	24

4.4	Boosts in DSR	27
4.4.1	Boost in a Generic Direction	27
4.4.2	Symmetrization of Deformed Boosts	30
4.4.3	Velocity Composition Law in $\tilde{M}$ and the Invariant Maximal Speed	32
4.4.4	Choosing the Boost Direction in DSR	34
4.4.5	Appendix — Another Derivation of a Generic Boost	35
5	Relativistic Kinematics in a Deformed Minkowski Space	37
6	Wave Propagation in a Deformed Space-Time	41
6.1	Deformed Helmholtz Equation	41
6.2	Propagation in an Undersized Waveguide	42
6.3	Total Internal Reflection	44
6.4	Deformation Tensor	48

III Metric Description of Fundamental Interactions51

7	Nonlocal Effects in Electromagnetic Interaction	53
7.1	Non-Minkowskian Metric for the E.M. Interaction	53
7.1.1	Photon Tunneling and Superluminal Velocities	53
7.1.2	Deformed E.M. Metric from the Cologne Experiment	55
7.2	An Application: Unified View to Cologne and Florence Experiments	59
7.2.1	Superluminal Propagation and the Friis Law	59
7.2.2	Two-Antenna System as a Barrier	64
8	Energy-Dependent Metric for Gravitation	67
8.1	Analysis of Experimental Data on Clock Rates	67
8.2	Local Metric of Gravitation	70
8.3	How Fast Do Travel Gravitational Effects?	72
8.4	Discussion of the Einstein versus Bohr Gedankenexperiment	73
8.5	Einstein and the Energy-Dependence of the Light Speed	75



9	Weak Interaction	77
10	Strong Interaction	81
10.1	The Bose–Einstein Correlation	81
10.2	DSR Treatment of BE Correlation	82
10.3	Deformed BE Correlation Function and the Strong Metric	85
10.4	Hadronic Time Deformation	89
10.5	A Geometric View to Confinement and Asymptotic Freedom	90
11	Metrics of Interactions	93
11.1	Review of Phenomenological Metrics	93
11.1.1	Energy-Dependent Phenomenological Metrics for the Four Interactions	93
11.1.2	Threshold Energies and Recursive Metrics	96
11.2	Asymptotic Metrics	97
IV	Breakdown of Local Lorentz Invariance	101
12	Experimental Tests of Local Lorentz Invariance	103
13	A New Electromagnetic Test of LLI	107
13.1	Theoretical Foundations	107
13.2	Experimental Setup and Measurement Procedure	110
13.3	Experimental Results	112
13.3.1	First Experiment	112
13.3.2	Second Experiment	113
13.3.3	An Interpretation by DSR	116
13.4	LLI Violation Parameter	118
13.5	Appendix — Einstein and the Aether	123
14	The Gravitational Mass of Electron from Geometry	125
14.1	LLI Breaking Factor and Relativistic Energy in DSR	125
14.2	The Electron as a Fundamental Particle and its “Geometrical” Mass	127

15	Toward the Fifth Dimension	131
15.1	From LLI Breakdown to Energy as Fifth Dimension	131
15.2	Embedding of Deformed Minkowski Space in a Five-Dimensional Riemannian Space	132
15.3	Einstein's Equations in Vacuum	134
	<i>References</i>	137
	<i>Index</i>	141

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PART I

GENERALIZING SPECIAL RELATIVITY

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CHAPTER 1

AN AXIOMATIC VIEW TO SPECIAL RELATIVITY

1.1. Foundations of Special Relativity

Special Relativity (SR) is essentially grounded on the properties of space-time, *i.e.* isotropy of space and homogeneity of space and time (as a consequence of the equivalence of inertial frames) and on the Galilei principle of relativity.

The two basic postulates of SR in its axiomatic formulation are¹:

1. *Space-time properties*: Space and time are homogeneous and space is isotropic.
2. *Principle of Relativity (PR)*: All physical laws must be covariant when passing from an inertial reference frame K to another frame K' , moving with constant velocity relative to K .

In the second postulate it is clearly understood that, for a correct formulation of SR, it is necessary to specify the total class, C_T , of the physical phenomena to which the relativity principle applies. The importance of such a specification is easily seen if one thinks that, from an axiomatic viewpoint, the only difference between Galilean and Einsteinian relativities just consists in the choice of C_T (*i.e.* the class of mechanical phenomena in the former case, and of mechanical and electromagnetic phenomena in the latter).

Such kinds of phenomena are ruled by fundamental interactions derivable from scalar (algebraic) or vector (Euclidean) potentials. Mechanical phenomena, corresponding to the class C_T of Galilean relativity, are determined by a uniform and stationary gravitational field (fully described in terms of a Newtonian potential) whereas electromagnetic phenomena considered within SR are determined by the potential component of the electromagnetic interaction.

Depending on the explicit choice of C_T , one gets *a priori* different realizations of the theory of relativity (in its abstract sense), each one

embedded in the previous. Of course, the principle of relativity, together with the specification of the total class of phenomena considered, necessarily implies, for consistency, the uniqueness of the transformation equations connecting inertial reference frames.

It is possible to show that, from the above two postulates, there follow — without any additional hypothesis — all the usual “principles” of SR, *i.e.* the “principle of reciprocity”, the linearity of transformations between inertial frames, and the invariance of light speed in vacuum.¹

Concerning this last point, it can be shown in general that postulates 1 and 2 above imply the existence of an invariant, real quantity, having the dimensions of the square of a speed, whose value must be experimentally determined in the framework of the total class C_T of the physical phenomena.^a Such an invariant speed depends on the interaction (fundamental, or at least phenomenological) ruling the physical phenomenon considered. Therefore *there is, a priori, an invariant speed for every interaction*, namely, a maximal causal speed for every interaction.

All the formal machinery of SR in the Einsteinian sense (including Lorentz transformations and their implications, and the metric structure of space-time) is simply a consequence of the above two postulates and of the choice, for the total class of physical phenomena C_T , of the class of mechanical and electromagnetic phenomena.

1.2. The Principle of Solidarity

The attempt at including the class of nuclear and subnuclear phenomena in the total class of phenomena for which special relativity holds true is therefore expected to imply a generalization of Minkowski metric, analogously to the generalization from the Euclidean to the Minkowski metric in going from mechanics to electrodynamics.

However, in order to avoid misunderstandings, it must be stressed that such an analogy with the extension of the Euclidean metric has to be understood not in the purely geometric meaning, but rather in the sense (already stressed by Penrose²) of Euclidean geometry as a physical theory. We shall discuss such a point in more detail in Chap. 3.

Indeed, the generalized metric is endowed with a dynamical character and is not only a consequence, but also an effective description of (the interaction involved in) the class of phenomena considered. It implements

^aThe invariant speed is obviously ∞ for Galilei’s relativity, and c (light speed in vacuum) for Einstein’s relativity.

therefore a feedback between interactions and space-time structure, already realized for gravitation in General Relativity.

This complies with the “*Principle of Solidarity*” stated by B. Finzi in the form already quoted in the Preface, which can be embodied in the following third principle of Relativity:

3. *Principle of Solidarity (PS)*: Each class of phenomena (namely, each interaction) determines its own space-time.

The mathematical tool to implement the PS is a *deformation* (in the sense specified later on) of the Minkowski metric. This implies, among the others, new, generalized transformation laws, which admit, as a suitable limit, the Lorentz transformations (just like Lorentz transformations represent a covering of the Galilei–Newton transformations).

1.3. Locality and Interaction

1.3.1. *Locality and Nonlocality*

The Principle of Solidarity can in general apply also to interactions nonlocal and not derivable from a potential. Let us therefore clarify what it is meant by such terms in this book.

Essentially two definitions of nonlocality exist in literature. The first amounts to contradict the so-called *Einstein–Bell locality*, which can be stated as follows:

The elements of physical reality of a system cannot be affected instantaneously at a distance (Einstein)

or

The probability of two measurements performed on events separated by a spacelike interval is simply the product of the probabilities of the two measurements separately (Bell).

It is easy to see that such a nonlocality (of quantum nature) is basically connected to the possibility of superluminal signals.

The second definition is related to the functional dependence of the force. A force is local when it depends on a space-time *point* (or, better, on an *infinitesimal* neighborhood of the point); it is nonlocal when it depends on a whole (*finite*) space-time *region*. In the following, we shall just mean this latter definition, whenever using the term nonlocal.



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1.3.2. Potential and Nonpotential Interactions

Let us stress, however, that “local” interaction (in the sense specified above) and “potential” interaction are *not* synonymous, in general. Once one fixes a space-time point, a local interaction is uniquely determined by an infinitesimal neighborhood of the point, whereas a potential interaction is just determined by the value the potential function takes at the point considered. Notice that the derivability from a potential requires the uniqueness of the potential function on the whole space-time region where the force field is defined.

An example of a nonlocal but potential interaction is provided by an interaction described by the potential

$$V(x_i) = \int \prod_{x_j \in \{x\}, j \neq i} dx_j V(\{x\}) \quad (1.1)$$

where $\{x\}$ is the set of metric coordinates, the integration can be definite or indefinite (in the latter case, the potential will depend also on the geometry of the integration regions) and $V(\{x\})$ is regular enough to ensure its integrability (for instance, in the Riemann sense).

On the other side, the electromagnetic (e.m.) interaction associated to a magnetic monopole is an example of a local but nonpotential interaction. In this case, due to the presence of the singular Dirac string, the force field of the monopole is irrotational locally but not globally.^b This implies that the monopole field is described by *many* (in general different) *local potentials*. By the non-uniqueness of the potential, the e.m. interaction of the magnetic monopole is nonpotential, but it is local indeed.

$\dots, \theta_1, \theta_2, \theta_3, \dots, \theta_{n-1}, \theta_n, \dots$

^bIn the language of differential geometry, the field of a Dirac magnetic monopole is associated to a differential form which is closed but not globally exact

CHAPTER 2

DEFORMED MINKOWSKI SPACE-TIME

As is well known, the Minkowski metric^a

$$g = \text{diag}(1, -1, -1, -1) \quad (2.1)$$

is a generalization of the Euclidean metric $\epsilon = \text{diag}(1, 1, 1)$. On the basis of the discussion of Chap. 1, we assume that the metric describes, in an effective way, the interaction, and that there exist interactions more general than the electromagnetic ones (which, as well known, are long-range and derivable from a potential).

The simplest generalization of the space-time metric which accounts for such more general properties of interactions is provided by a *deformation*, η , of the Minkowski metric (2.1), defined as³⁻⁵

$$\eta = \text{diag}(b_0^2, -b_1^2, -b_2^2, -b_3^2). \quad (2.2)$$

Of course, from a formal point of view metric (2.2) is not new at all. Deformed Minkowski metrics of the same type have already been proposed in the past⁶⁻⁹ in various physical frameworks, starting from Finsler's generalization of Riemannian geometry⁶ to Bogoslawsky's anisotropic space-time⁷ to Santilli's isotopic Minkowski space.⁸ A phenomenological deformation of the type (2.2) was also obtained by Nielsen and Picek⁹ in the context of the electroweak theory. Moreover, although for quite different purposes, "quantum" deformed Minkowski spaces have been also considered in the context of quantum groups.¹⁰ Leaving to later considerations the true specification of the exact meaning of the deformed metric (2.2) in our framework, let us right now stress two basic points.

^aThroughout this book, lower Latin indices take the values $\{1, 2, 3\}$ and label spatial dimensions, whereas lower Greek indices vary in the range $\{0, 1, 2, 3\}$, with 0 referring to the time dimension. Ordinary 3-vectors are denoted in boldface. Moreover, we adopt the signature $(+, -, -, -)$, and employ the notation "*ESCon*" ("*ESCoff*") to mean that the Einstein sum convention on repeated indices is (is not) used.

1. Firstly, metric (2.2) is supposed to hold at a *local* (and not global) scale, *i.e.* to be valid not everywhere, but only in a suitable (local) space-time region (characteristic of both the system and the interaction considered). We shall therefore refer often to it as a “*topical*” deformed metric.

In the present case, the term “local” must be understood in the sense that a deformed metric of the kind (2.2) describes the geometry of a 4-dimensional variety attached at a point x of the standard Minkowski space-time, in the same way as a local Lorentz frame is associated (as a tangent space) to each point of the (globally Riemannian) space of Einstein’s GR. Another example, on some respects more similar to the present formalism, is provided by a space-time endowed with a vector fibre-bundle structure, where a (maximally symmetric) Riemann space with constant curvature is attached at each point x .¹¹

2. Secondly, metric (2.2) is regarded to play a *dynamical role*. So, in order to comply with the solidarity principle, we assume that the parameters b_μ ($\mu = 0, 1, 2, 3$) are, in general, real and positive functions of a given set of observables $\{\mathcal{O}\}$ characterizing the system (in particular, of its total energy, as specified later):

$$\{b_\mu\} = \{b_\mu(\{\mathcal{O}\})\} \in R_0^+, \quad \forall \{\mathcal{O}\}. \quad (2.3)$$

The set $\{\mathcal{O}\}$ represents therefore, in general, a set of non-metric variables ($\{x_{n.m.}\}$).

Equation (2.2) therefore becomes:

$$\begin{aligned} \eta_{\mu\nu} &= \eta_{\mu\nu}(\{\mathcal{O}\}) \\ &= \text{diag}(b_0^2(\{\mathcal{O}\}), -b_1^2(\{\mathcal{O}\}), -b_2^2(\{\mathcal{O}\}), -b_3^2(\{\mathcal{O}\})) \stackrel{\text{ESC}}{=}^{\text{off}} \\ &= \delta_{\mu\nu}(\delta_{\mu 0}b_0^2(\{\mathcal{O}\}) - \delta_{\mu 1}b_1^2(\{\mathcal{O}\}) - \delta_{\mu 2}b_2^2(\{\mathcal{O}\}) - \delta_{\mu 3}b_3^2(\{\mathcal{O}\})). \end{aligned} \quad (2.4)$$

However, for the moment the deformation of the Minkowski space will be discussed only from a formal point of view, by disregarding the problem of the observables on which the coefficients b_μ actually depend (it will be faced in Chap. 3).

Notice that the first point, *i.e.* the assumed local validity of (2.2), differentiates this approach from those based on Finsler’s geometry or from the Bogoslawski’s one (which, at least in their standard meaning, do consider deformed metrics at a *global* scale), and makes it similar, on some aspects, to the philosophy and methods of the isotopic generalizations of Minkowski spaces. However, it is well known that Lie-isotopic theories rely in an essential way, from the mathematical standpoint, on (and are strictly characterized by) the very existence of the so-called isotopic unit.⁸ In the

following, such a formal device will not be exploited (because unessential on all respects), so that the present formalism is not an isotopic one. Moreover, from a physical point of view, the isotopic formalism is expected to apply only to strong interactions.⁸ On the contrary, it will be assumed that the (effective) representation of interactions through the deformed metric (2.2) does hold for *all* kinds of interactions (at least for their nonlocal component). In spite of such basic differences this formalism shares some common formal results — as we shall see in the following — with isotopic relativity (like the mathematical expression of the generalized Lorentz transformations).

It is now possible to define a generalized (“*deformed*”) Minkowski space $\tilde{M}(x, \eta(\{\mathcal{O}\}))$ with the same local coordinates x of M (the four-vectors of the usual Minkowski space), but with metric given by the metric tensor η (2.4). The generalized interval in $\tilde{M}$ is therefore given by $(x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$, with c being the usual light speed in vacuum) (ESC on)³⁻⁵:

$$\begin{aligned} ds^{\tilde{2}}(\{\mathcal{O}\}) &= b_0^2(\{\mathcal{O}\})c^2dt^2 - b_1^2(\{\mathcal{O}\})(dx^1)^2 - b_2^2(\{\mathcal{O}\})(dx^2)^2 - b_3^2(\{\mathcal{O}\})(dx^3)^2 \\ &= \eta_{\mu\nu}(\{\mathcal{O}\})dx^\mu dx^\nu = dx * dx. \end{aligned} \quad (2.5)$$

The last step in (2.5) defines the scalar product $*$ in the deformed Minkowski space $\tilde{M}$. Moreover, according to (2.5), we shall use the following notation for the deformed square norm of a four-vector:

$$|x|_*^2 \equiv x * x = \eta_{\mu\nu}(\{\mathcal{O}\})x^\mu x^\nu = x^{\tilde{2}}. \quad (2.5')$$

In order to evidence some preliminary implications of metric (2.4), let us consider (for simplicity sake and without loss of generality) an isotropic 3-dimensional space, *i.e.*

$$b_1^2(\{\mathcal{O}\}) = b_2^2(\{\mathcal{O}\}) = b_3^2(\{\mathcal{O}\}) \equiv b^2(\{\mathcal{O}\}) \quad (2.6)$$

so that the corresponding deformed metric reads

$$\begin{aligned} \eta_{\mu\nu\text{ISO}}(\{\mathcal{O}\}) &= \text{diag}(b_0^2(\{\mathcal{O}\}), -b^2(\{\mathcal{O}\}), -b^2(\{\mathcal{O}\}), -b^2(\{\mathcal{O}\})) \stackrel{\text{ESC}}{=}^{\text{off}} \\ &= \delta_{\mu\nu}[\delta_{\mu 0}b^2(\{\mathcal{O}\}) - (\delta_{\mu 1} + \delta_{\mu 1} + \delta_{\mu 1})b^2(\{\mathcal{O}\})]. \end{aligned} \quad (2.7)$$

One gets, for null separation $ds^{\tilde{2}} = 0$:

$$\begin{aligned} ds^{\tilde{2}}(\{\mathcal{O}\}) = 0 &\Leftrightarrow b_0^2(\{\mathcal{O}\})c^2dt^2 - b^2(\{\mathcal{O}\})[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] = 0 \\ &\Leftrightarrow \frac{(dx^2 + dy^2 + dz^2)}{dt^2} = c^2 \frac{b_0^2(\{\mathcal{O}\})}{b^2(\{\mathcal{O}\})}. \end{aligned}$$



From Eq. (2.8) it is easily seen that the *maximal causal speed* in the generalized Minkowski space is given by:

$$u = \frac{b_0(\{\mathcal{O}\})}{b(\{\mathcal{O}\})} c. \quad (2.9)$$

It is worth noticing that a similar result (namely, a “maximum attainable speed”, *a priori* different for different physical processes) was also obtained by Coleman and Glashow,¹² in the framework of a discussion of possible effects breaking Lorentz invariance (essentially on a local scale).

Let us remark that u depends explicitly on the metric parameters b_μ , which are *a priori* different for every physical system. However, since the deformation of the metric represents, on average, the effects of the nonlocal interactions involved, it is expected that physical systems with the same kind of interactions (besides the electromagnetic ones) are described by metric parameters of the same order of magnitude (or, at least, this holds true for the ratio b_0/b). In this sense it is possible to refer to u as a “speed of interaction”, rather than “speed of the physical system” considered (of course, at the same energy scale).

In Eq. (2.9), the value of u is parametrized in terms of c , and depends on the physical system (and its interactions). Moreover, it is

$$u \geq c \Leftrightarrow \frac{b_0(\{\mathcal{O}\})}{b(\{\mathcal{O}\})} \geq 1. \quad (2.10)$$

In other words, there may be maximal causal speeds either *subluminal* or *superluminal*, depending on the interaction considered.

It is worth to recall that the deformation of the metric, resulting in the interval (2.5), represents a geometrization of a suitable space-time region (corresponding to the physical system considered) that describes, in the average, the effect of nonlocal interactions on a test particle. It is clear that there exist infinitely many deformations of the Minkowski space (precisely, ∞^4), corresponding to the different possible choices of the parameters b_μ , *a priori* different for each physical system.

Moreover, since the usual, “flat” Minkowski metric g (2.1) is related in an essential way to the electromagnetic interaction, we shall always mean in the following — unless otherwise specified — that electromagnetic interactions imply the presence of a fully Minkowskian metric. Actually, as it will be seen, a deformed metric of the type (2.7) is required if one wants to account for possible nonlocal electromagnetic effects (like those occurring in the superluminal wave propagation in waveguides: see Chap. 7).

CHAPTER 3

DESCRIPTION OF INTERACTIONS BY ENERGY-DEPENDENT METRICS

3.1. Energy and the Finzi Principle

Now one has to go into the question of the dependence of the metric parameters $b_\mu(\{\mathcal{O}\})$ on the set of observables $\{\mathcal{O}\}$ of the system considered, and to examine closely the physical meaning of such a functional dependence.

To this aim, the basic question to be put is how to implement Finzi's Principle of Solidarity for all interactions on a mere special-relativistic basis. At present, General Relativity (GR) is the only successful theoretical realization of geometrizing an interaction (the gravitational one). As is well known, energy plays a fundamental role in GR, since the energy-momentum tensor of a given system is the very source of the gravitational field.

A moment's thought shows that this occurs actually also for other interactions. Let us consider, for instance, the case of Euclidean geometry in its intrinsic meaning of a theory of the physical reality at its basic classical (macroscopic) level. Actually, it describes in a quantitative way, in mathematical language, the relations among measured physical entities — distances, in this case —, and therefore of the physical space in which phenomena occur.

However, the measurement of distances depends on the motion of the body which actually performs the measurement. Such a dependence is indeed not on the *kind* of motion, but rather on the *energy* needed to let the body move, and on the *interaction* providing such energy. The measurement of time needs as well a periodic motion with constant frequency, and therefore it too depends on the energy and on the interaction.

This simple example shows how *energy does play a fundamental role in determining the very geometrical structure of space-time*.

Generalizing such an argument, we can state that exchanging energy between particles amounts to measure operationally their space-time

separation.^a Of course such a process depends on the interaction involved in the energy exchange; moreover, each exchange occurs at the maximal causal speed characteristic of the given interaction. It is therefore natural to assume that the measurement of distances, performed by the energy exchange according to a given interaction, realizes the “solidarity principle” between space-time and interactions at the microscopic scale. This allows one to identify the total energy E of the physical process considered as the observable on which the coefficients $b_\mu(\{\mathcal{O}\})$ depend:

$$\{\mathcal{O}\} \equiv E \Leftrightarrow \{b_\mu(\{\mathcal{O}\})\} \equiv \{b_\mu(E)\}, \quad \forall \mu = 0, 1, 2, 3. \quad (3.1)$$

Actually, since all the functions $\{b_\mu\}$ are dimensionless, they must depend on a dimensionless variable. Then, one has to divide the energy E by a constant E_0 (in general characteristic of each fundamental interaction), with dimensions of energy, so that:

$$\{b_\mu(\{\mathcal{O}\})\} \equiv \left\{ b_\mu \left(\frac{E}{E_0} \right) \right\}, \quad \forall \mu = 0, 1, 2, 3. \quad (3.2)$$

As it will be seen, E_0 has the meaning of a “threshold energy”.

Thus, the distance measurement is accomplished by means of the deformed metric tensor, given explicitly by

$$\begin{aligned} \eta_{\mu\nu}(E) &= \text{diag}(b_0^2(E), -b_1^2(E), -b_2^2(E), -b_3^2(E)) \stackrel{\text{ESC}}{=}^{\text{off}} \\ &= \delta_{\mu\nu}(\delta_{\mu 0} b_0^2(E) - \delta_{\mu 1} b_1^2(E) - \delta_{\mu 2} b_2^2(E) - \delta_{\mu 3} b_3^2(E)). \end{aligned} \quad (3.3)$$

Any interaction can be therefore phenomenologically described by metric (3.3) in an *effective* way. This is true in general, but necessary in the case of nonlocal and nonpotential interactions. For force fields which admit a potential, such a description is complementary to the actual one.^b

One is therefore led to put forward a revision of the concept of “geometrization of an interaction”: each interaction produces its own metric, formally expressed by the metric tensor (3.3), but realized via different choices of the set of parameters $b_\mu(E)$. Otherwise said, the $b_\mu(E)$ ’s are peculiar to every given interaction. The statement that (3.3) provides us with a metric description of an interaction must be just understood in such a sense.

^aNotice that, in this framework, a space-time point has only a mathematical (geometrical) meaning, since it physically corresponds to an energy insufficient to the motion (for the interaction considered). See also Sec. 11.2.

^bAs we shall see in Chap. 8, an example is just provided by the gravitational interaction in the Newtonian limit.

Therefore, the energy-dependent deformation of the Minkowski metric implements a generalization of the concept of geometrization of an interaction (in accordance with Finzi's principle). The GR theory implements a geometrization (at a *global* scale) of the gravitational interaction, based on its derivability from a potential and on the equivalence between the inertial mass of a body and its "gravitational charge". The formalism of energy-dependent metrics allows one instead to implement a geometrization (at a *local* scale) of any kind of interaction characterized by a phenomenology experimentally measurable. As already stressed before, such a formalism applies, in principle, to both fundamental and phenomenological interactions, either potential (gravitational, electromagnetic) or nonpotential (strong, weak), *local* and *nonlocal* (in the sense already specified), for which either an Equivalence Principle holds (as it is the case of gravitation) or (in the more general case) the inertial mass of the body *is not* in general proportional to its charge in the force field considered (e.m., strong, and weak interaction). The basic point of the present way of geometrizing an interaction (thus realizing the Finzi legacy) consists in a "upsetting" of the space-time-energy parametrization. Whereas for potential interactions there exists a potential energy depending on the space-time metric coordinates, one has here to do with a deformed metric tensor η , whose coefficients depend on the energy, that thus assumes a *dynamical* role.

3.2. Energy as Dynamical Variable

From the physical point of view, the energy E is to be understood as the measured energy of the system, and must be therefore regarded as a merely phenomenological variable. As is well known, all the physically realizable detectors work via their electromagnetic interaction in the usual Minkowski space. This is why, in this formalism, the Minkowski space and the e.m. interaction do play a fundamental role. The former is, as already stressed, the cornerstone on which to build up the generalization of Special Relativity based on the deformed metric (3.3). The latter is the comparison term for all fundamental interactions. Let us recall that they are strictly interrelated, since it is just electromagnetism which determines the Minkowski geometry.

From the mathematical standpoint, E is to be considered as a dynamical variable, because it specifies the dynamical behavior of the process under consideration, and, via the metric coefficients, it provides us with a dynamical map — in the energy range of interest — of the interaction ruling the given process.

Let us notice that metric (3.3) plays, for nonpotential interactions, a role analogous to that of the Hamiltonian H for a potential interaction. In particular, the metric tensor η as well is not an input of the theory, but must be built up from the experimental knowledge of the physical data of the system concerned (in analogy with the specification of the Hamiltonian of a potential system). However, there are some differences between η and H worth to be stressed. Indeed, as is well known, H represents the total energy E_{tot} of the system irrespective of the value of E_{tot} and the choice of the variables. On the contrary, $\eta(E)$ describes the variation in the measurements of space and time, in the physical system considered, as E_{tot} changes; therefore, η does depend on the numerical value of H , but not on its functional form. The explicit expression of η depends only on the interaction involved.

It is moreover worth recalling that the use of an energy-dependent space-time metric can be traced back to Einstein himself, who generalized the Minkowski interval as follows

$$ds^2 = \left(1 + \frac{2\phi}{c^2}\right) c^2 dt^2 - (dx^2 + dy^2 + dz^2) \quad (3.4)$$

(where ϕ is the Newtonian gravitational potential), in order to account for the modified rate of a clock in presence of a (weak) gravitational field.

One may be puzzled about the dependence of the metric on the energy, which is not an invariant under usual Lorentz transformations, but transforms like the time-component of a four vector.

Actually, energy is to be regarded, in this formalism, from two different points of view. One has, on one side, the energy as measured in full Minkowskian conditions, which, as such, behaves as a genuine four-vector under usual Lorentz transformations (in the sense that it changes in the usual way if we go, say, from the laboratory frame to another frame in uniform motion with respect to it). Once fixed the frame, one gets a measured value of the energy for a given process (for instance, the energy of the Bose-Einstein correlation phenomenon in pion production, as measured at CERN by the UA1 collaboration). This is the value which enters, as a parameter, in the expression (3.3) of the deformed metric. Such an energy, therefore, is no longer to be considered as a four vector in the deformed Minkowski space, but it is just a quantity whose value determines the deformed geometry of the process considered (or, otherwise speaking, which selects the deformed space-time we have to use to describe the phenomenon).

Let us briefly discuss the phenomenological aspects of the metric dependence on energy. Notice that, in particle collisions, the energy of the incident

particle in the laboratory frame, $E = E_L$, can be related to the Mandelstam invariant s (corresponding to the total energy of the colliding particles in the center-of-mass reference frame) by means of the relation, valid at sufficiently high energies:

$$s \approx 2E_L M, \quad (3.5)$$

(where M is the mass of the target),^c and analogous to the formula relating the laboratory momentum p_L and the invariant flux ϕ : $\phi = p_L M$.

Therefore, in particle collisions, the metric parameters can be indeed considered as dependent on the invariant s :

$$\{b_\mu(E_L)\} \approx \left\{b_\mu\left(\frac{s}{2M}\right)\right\} \equiv \{\tilde{b}_\mu(s)\}. \quad (3.6)$$

Actually, s is a scalar under usual Lorentz transformations, and, in general, an usual relativistic invariant is no longer unchanged for transformations preserving the deformed interval (2.5). However, let us recall that the energy chosen as phenomenological parameter is that measured electromagnetically, and therefore in presence of a Minkowski metric. This interpretation is supported by the case of colliding beam reactions with different energies: indeed, it would be impossible, otherwise, to define what energy must be used as parameter. Such a point of view can be adopted e.g. in the analysis of the so called “ramping run” of UA1 in order to extract the parameters of the hadronic metric (as functions of the energy) from the experimental data (see Part III).

^cIndeed, the Mandelstam invariant s for two colliding particles is defined as $s \equiv (p_1 + p_2)^2$, where p_i is the 4-momentum of the i th particle. In their C.M. frame, $\underline{p}_1 + \underline{p}_2 = 0$, and $s = (E_1 + E_2)^2$. In the laboratory frame, where the target particle 2 of mass M is at rest, the 4-momenta of the two particles are:

$$p_1^\mu = \left(\frac{E_1}{c} = \sqrt{|\mathbf{p}_1|^2 + m^2 c^2}, \mathbf{p}_1\right), \quad p_2^\mu = (Mc, \mathbf{0}).$$

Therefore

$$\begin{aligned} s &= \frac{E_1^2}{c^2} + M^2 c^2 + 2E_1 M - |\mathbf{p}_1|^2 = |\mathbf{p}_1|^2 + m^2 c^2 + M^2 c^2 + 2E_1 M - |\underline{p}_1|^2 \\ &= m^2 c^2 + M^2 c^2 + 2E_1 M. \end{aligned}$$

In the (ultra) relativistic limit $E_1 \gg mc^2$, $E_1 \gg Mc^2$, we get therefore

$$s \approx 2E_1 M$$

which coincides with Eq. (3.5) on account of the fact that $E_1 \equiv E_L$.

In other cases, the phenomenological energy parameter may not be so easy to identify, neither it can be directly related to an invariant quantity. This is the case, for instance, of the lifetime of unstable particles.

Let us explicitly stress that the theory of SR based on metric (2.4) has nothing to do with General Relativity. Indeed, in spite of the formal similarity between the interval (2.5), with the b_μ functions of the coordinates, and the metric structure of a Riemann space, in this framework no mention at all is made of the equivalence principle between mass and inertia, and among non-inertial, accelerated frames. Moreover, General Relativity describes geometrization on a large-scale basis, whereas the special relativity with topical deformed metric describes local (small-scale) deformations of the metric structure (although the term “small scale” must be referred to the real dimensions of the physical system considered).

But the basic difference is provided by the fact that actually the deformed Minkowski space $\tilde{M}$ has zero curvature, as it is easily seen by remembering that, in a Riemann space, the scalar curvature is constructed from the derivatives, with respect to space-time coordinates, of the metric tensor. In others words, the space $\tilde{M}$ is *intrinsically flat* — at least in a mathematical sense. Namely, it would be possible, in principle, to find a change of coordinates, or a rescaling of the lengths, so as to recover the usual Minkowski space. However, such a possibility is only a mathematical, and not a physical one. This is related to the fact that the energy of the process is fixed, and cannot be changed at will. For that value of the energy, the metric coefficients do possess values different from unity, so that the corresponding space $\tilde{M}$, for the given energy value, is actually different from the Minkowski one. The usual space-time M is recovered for a special value E_0 of the energy (characteristic of any interaction), such that indeed

$$\eta(E_0) = g = \text{diag}(1, -1, -1, -1). \quad (3.7)$$

Such a value E_0 (which must be derived from the phenomenology) will be referred to as *the threshold energy of the interaction considered*. As we shall see, it is just the constant introduced in Eq. (3.2) by dimensional arguments.

It is worth stressing that, due to the dependence on energy of the metric parameters b_μ , the maximal causal speed of any interaction, too, is a function of the energy, according to the relation

$$u(E) = \frac{b_0(E)}{b(E)} c. \quad (3.8)$$

However, it remains invariant — for fixed energy values — under generalized Lorentz transformations from a given reference frame to another. Clearly,

in Eq. (3.8), the light speed in vacuum, c , does merely play the role of a phenomenological parameter on which the value of u depends.

The maximal causal speed u can be interpreted, from a physical standpoint, as the speed of the quanta of the interaction which requires a representation in terms of a generalized Minkowski space. Since these quanta are associated to lightlike world-lines in $\tilde{M}$ (see Eq. (2.8)), they must be zero-mass particles (with respect to the interaction considered), in analogy with photons (with respect to the e.m. interaction) in the usual SR.^d

Let us clarify the latter statement. The carriers of a given interaction propagating with the speed u typical of that interaction *are expected to be strictly massless only inside the space whose metric is determined by the interaction considered*. *A priori*, nothing forbids that such “deformed photons” may acquire a nonvanishing mass in a deformed Minkowski space related to a different interaction.

This might be the case of the massive bosons W^+ , W^- and Z^0 , carriers of the weak interaction, which would therefore be massless in the space $\tilde{M}(\eta_{\text{weak}}(E))$ related to the weak interaction, but would acquire a mass when considered in the standard Minkowski space M of SR (that, as already stressed, is strictly connected to the electromagnetic interaction,^e ruling the operation of the measuring devices). In this framework, therefore, it is not necessary to postulate a “symmetry breaking” mechanism (like the Goldstone one in gauge theories) to let particles acquire mass.^f Mass itself would assume a *relative nature*, related not only to the interaction concerned, but also to the metric background where one measures the energy of the physical system considered. This can be seen if one takes into account the fact in general, for relativistic particles, mass is the invariant norm of 4-momentum, and what is usually measured is not the value of such an invariant, but of the related energy. As it will be seen in Chap. 14, it is possible indeed, in this framework, to give a geometrical meaning to the electron mass, and relate it to the breakdown of local Lorentz invariance.

^dHowever, the problem of mass requires to be considered in more detail (see Chap. 14).

^eActually, as we shall see in Part III, this is strictly true only for $E > E_{0,\text{e.m.}}$, where $E_{0,\text{e.m.}} \simeq 4.5 \mu\text{eV}$ is the threshold energy for the e.m. interaction, i.e.

$$\tilde{M}(g_{\text{e.m.}}(E))|_{E \gtrsim 4.5 \mu\text{eV}} = M.$$

^fOn the contrary, if one could build up measuring devices based on interactions different from the e.m. one, the photon might acquire a mass with respect to such a non-e.m. background.

To end this chapter, let us notice that the deformation (3.3) of the Minkowski metric is expected to apply to the description not only of extended particles, but also of quantum pointlike particles, as far as their energy is such that one cannot neglect their associated cloud of virtual quanta.

The problem of a metric description of a given interaction is thus formally reduced to the determination of the coefficients $b_\mu(E)$ from the data on some physical system, whose dynamical behaviour is ruled by the interaction considered.

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PART II

**RELATIVITY IN A DEFORMED
SPACE-TIME**

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CHAPTER 4

GENERALIZED PRINCIPLE OF RELATIVITY AND LORENTZ TRANSFORMATIONS

4.1. Deformed Special Relativity

In order to develop the relativity theory in a deformed Minkowski space-time, one has to suitably generalize and clarify the basic concepts which are at the very foundation of SR.

Let us first of all define a “topical inertial frame”:

- (i) a *topical “inertial” frame* (TIF) is a reference frame in which space-time is homogeneous, but space is not necessarily isotropic.

Then, a “*generalized principle of relativity*” (or “*principle of metric invariance*”) can be stated as follows:

- (ii) all physical measurements within every topical “inertial” frame must be carried out via the *same* metric.

We shall call “*Deformed Special Relativity*” (DSR) the generalization of SR based on the above two postulates, and whose space-time structure is given by the deformed Minkowski space $\tilde{M}$ introduced in Chap. 2.

4.2. Deriving the Deformed Lorentz Transformations

It follows from the above points (i), (ii) that the transformation equations connecting topical “inertial” frames, *i.e.* the generalized Lorentz transformations, are those which leave invariant the deformed metric when passing from a topical “inertial” frame K , to another frame K' , moving with constant velocity with respect to K . Then, physical laws are to be covariant with respect to such generalized transformations.

In other words, the generalized Lorentz transformations are the isometries of the deformed Minkowski space $\tilde{M}$. We shall refer to them in the following as *deformed Lorentz transformations* (DLT). Their explicit form can be derived by the same procedure followed in order to find the Lorentz transformations in the usual Minkowski space.

Consider two TIF, K and K' ; by definition, the DLT's leave invariant the deformed interval (2.5), *i.e.*

$$b_0^2 c^2 t^2 - b_1^2 x^2 - b_2^2 y^2 - b_3^2 z^2 = b_0^2 c^2 t'^2 - b_1^2 x'^2 - b_2^2 y'^2 - b_3^2 z'^2. \quad (4.1)$$

Moreover, it can be assumed, without loss of generality, that the frames K and K' are in standard configuration (*i.e.* their spatial frames coincide at $t = t' = 0$). By choosing the boost direction along $\hat{x}^1 = \hat{x}$, we have therefore $y' = y$, $z' = z$ and Eq. (4.1) reduces to

$$b_0^2 c^2 t^2 - b_1^2 x^2 = b_0^2 c^2 t'^2 - b_1^2 x'^2. \quad (4.2)$$

From space-time homogeneity it follows that the functional relations between the two sets of coordinates $\{x, y, z, t\}$ and $\{x', y', z', t'\}$ must be linear. Then, in general, the deformed coordinate transformations are to be searched in the form

$$\begin{cases} x' = A_{11}x + A_{14}t \\ y' = y \\ z' = z \\ t' = A_{41}x + A_{44}t \end{cases} \quad (4.3)$$

where the coefficients $A_{11}, A_{14}, A_{41}, A_{44}$ depend *a priori* in general on $\mathbf{v}$ and $\hat{x}$ (and, parametrically, on the energy).

Notice that the origin O' of TIF K' must move in K with velocity $\mathbf{v} = v^1 \hat{x}$, and therefore:

$$x' = 0, \quad x = vt \Leftrightarrow A_{14} = -vA_{11} \Leftrightarrow x' = A_{11}(x - vt). \quad (4.4)$$

Replacing (4.3), (4.4) in (4.2) yields

$$b_0^2 c^2 t^2 - b_1^2 x^2 = b_0^2 c^2 (A_{41}x + A_{44}t)^2 - A_{11}^2 b_1^2 x^2 (x - vt)^2 \quad (4.5)$$

which implies the following 3×3 quadratic system:

$$\begin{cases} c^2 = c^2 A_{44}^2 - \left(\frac{b_1}{b_0}\right)^2 A_{11}^2 v^2 \\ -1 = c^2 \left(\frac{b_0}{b_1}\right)^2 A_{41}^2 - A_{11}^2 \\ 0 = c^2 \left(\frac{b_0}{b_1}\right)^2 A_{41} A_{44} + A_{11}^2 v \end{cases} \quad (4.6)$$



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with general solution

$$A_{11} = A_{44} = \pm \left(1 - \left(\frac{vb_1}{cb_0} \right)^2 \right)^{-1/2} \quad (4.7)$$

$$A_{41} = \mp \left(\frac{vb_1^2}{c^2 b_0^2} \right) \left(1 - \left(\frac{vb_1}{cb_0} \right)^2 \right)^{-1/2} = - \left(\frac{vb_1^2}{c^2 b_0^2} \right) A_{11}. \quad (4.8)$$

The final result is

$$\begin{cases} x' = \tilde{\gamma}(x - vt) = \tilde{\gamma} \left(x - \tilde{\beta} \frac{b_0}{b_1} ct \right) \\ y' = y \\ z' = z \\ t' = \tilde{\gamma} \left(t - \frac{vb_1^2}{c^2 b_0^2} x \right) = \tilde{\gamma} \left(t - \frac{\tilde{\beta}^2}{v} x \right) \end{cases} \quad (4.9)$$

where v is the relative speed of the reference frames, and

$$\tilde{\beta} = \frac{v}{u}; \quad (4.10)$$

$$\tilde{\gamma} = (1 - \tilde{\beta}^2)^{-1/2}. \quad (4.11)$$

Transformations (4.9) do formally coincide with the isotopic Lorentz transformations. However, in the present context their physical meaning is different, as it is easily seen e.g. by the identification of the maximal causal speed u with the speed characteristic of the quanta of a given interaction (see Sec. 3). In particular, the parametrization (4.10) of the deformed velocity parameter $\tilde{\beta}$ in terms of u immediately shows that is always $\tilde{\beta} < 1$, so that $\tilde{\gamma}$ never takes imaginary values (contrarily to the isotopic case). Moreover, no reference at all is made, in this framework, to the existence of an underlying “medium”.

It must be carefully noted that, like the metric, also the generalized Lorentz transformations depend on the energy. This means that one gets different transformation laws for different values of E , but still with the same functional dependence on the energy, so that the invariance of the deformed interval (2.5) is always ensured (provided that the process considered does always occur via the same interaction).

Indeed, the energy E can be considered fixed also because, from a quantum point of view, energy can be transferred only by **Documents**.

Differentiating Eqs. (4.9), we get therefore

$$\begin{cases} udt' + t'du = \tilde{\gamma}(udt - \tilde{\beta}dx) + [d\tilde{\gamma}(ut - \tilde{\beta}x) + \tilde{\gamma}(tdu - x d\tilde{\beta})]; \\ dx' = \tilde{\gamma}(dx - \tilde{\beta}udt) + [d\tilde{\gamma}(x - \tilde{\beta}ut) - \tilde{\gamma}(t\tilde{\beta}du + td\tilde{\beta})], \end{cases} \quad (4.12)$$

where, by the above argument, $dE = 0$ and therefore $d\tilde{\gamma} = d\tilde{\beta} = du = 0$. Squaring (4.12) and subtracting, we find

$$dx'^2 - u^2 dt'^2 = \tilde{\gamma}^2 [(dx - \tilde{\beta}dt)^2 - (udt - \tilde{\beta}dx)^2] = dx^2 - u^2 dt^2 \quad (4.13)$$

where in the last step use has been made of Eq. (4.2). Exploiting the explicit expression of u , Eq. (2.9), one has finally

$$ds'^2 = ds^2, \quad (4.14)$$

i.e. the deformed Lorentz transformations (4.9) are actually the isometries of the deformed Minkowski space $\tilde{M}$, in spite of their dependence on the energy.

As already stressed in Chap. 2 (see Eq. (2.10)), it is possible to have $u \geq c$ and therefore $c \leq v \leq u$, *i.e.* superluminal motions are allowed. Let us remark that the possibility of tachyonic speeds is accomplished, within this framework, without any recourse neither to imaginary quantities nor to singularities in the transformation laws (unlike the standard case), because it is always $v \leq u$ (even if $v \geq c$), so that the relativistic factor $\tilde{\gamma}$ (Eq. (4.11)) takes only real values, as already noted above.

4.3. Maximal Causal Speed Revisited

In $\tilde{M}$, it is possible *a priori* to consider two scalar products between 3-vectors $\mathbf{v}_1, \mathbf{v}_2$: the standard, Euclidean one $\cdot$, defined by means of the metric tensor $g_{ik} = \delta_{ik}$, and the deformed one, induced by the deformed scalar product $*$ in $\tilde{M}$, and defined by means of the metric tensor $-\eta_{ik}(E) \stackrel{\text{ESCOff}}{=} b_i^2(E)\delta_{ik}$ (where the sign $-$ is obviously introduced in order to get a positive 3-vector norm) as follows (cf. Eq. (2.5)):

$$\begin{aligned} \mathbf{v}_1 * \mathbf{v}_2 &\equiv - \sum_{i=1}^3 \eta_{ij}(E)(v_1)^i (v_2)^j \\ &= \sum_{i=1}^3 b_i^2(E)\delta_{ij}(v_1)^i (v_2)^j \\ &= b_1^2(E)(v_1)^1 (v_2)^1 + b_2^2(E)(v_1)^2 (v_2)^2 + b_3^2(E)(v_1)^3 (v_2)^3 \end{aligned} \quad (4.15)$$

In the following, $|\mathbf{v}|_*$ will denote the absolute value of a 3-vector with respect to the deformed scalar product $*$ (cf. Eq. (2.5')), whereas the notation $|\mathbf{v}| = v$ will be used for the norm of $\mathbf{v}$ with respect to the standard product $\cdot$.

As is well known, the maximal causal speed in M is obtained by putting $ds^2 = 0$, whence

$$ds^2 = 0 \Leftrightarrow c^2 dt^2 - dx^2 - dy^2 - dz^2 = 0 \Leftrightarrow \frac{dx^2 + dy^2 + dz^2}{dt^2} = c^2. \quad (4.16)$$

Then one interprets c as the maximal causal speed along any direction of the (Euclidean) space R^3 (embedded in the pseudo-euclidean Minkowski space-time M). Such an interpretation is obviously based on the physical fact that c coincides with the light speed in vacuum, and on the isotropy of R^3 . Therefore c represents the value of any of the three components of the maximal causal velocity vector (m.c.v.) of SR, $\mathbf{u}_{SR}$, namely:

$$\mathbf{u}_{SR} = (c, c, c). \quad (4.17)$$

Then, c^2 is not, in general, a square modulus, but the square of any component of $\mathbf{u}_{SR}$, whose square modulus (with respect to the Euclidean scalar product $\cdot$), is instead:

$$|\mathbf{u}_{SR}|^2 \equiv \sum_{i=1}^3 (u_{SR}^i)^2 = 3c^2 \quad (4.18)$$

so that

$$u_{SR}^i = \frac{1}{\sqrt{3}} |\mathbf{u}_{SR}| \quad \forall i = 1, 2, 3. \quad (4.19)$$

The above procedure must be suitably modified in the DSR case, due to the space anisotropy of $\tilde{M}$.

Actually, in order to sort out a single component of the 3-vector m.c.v., in a general 4-d special-relativistic theory (characterized by a diagonal metric tensor $g_{\mu\nu}(\{\mathcal{O}\})$, where $\{\mathcal{O}\}$ is a set of observables corresponding to non-metric variables), one has to exploit a “*directional separation*” (or “*dimensional separation*”) method, which consists of the following three-step recipe (ESCoFF throughout):

1. Set $ds^{\tilde{2}}$ equal to zero:

$$ds^{\tilde{2}} = 0 \Leftrightarrow g_{00}(\{\mathcal{O}\})c^2 dt^2 + \sum_{i=1}^3 g_{ii}(\{\mathcal{O}\})(dx^i)^2 = 0 \quad (4.20)$$

2. In order to find the i th component $u^i(\{\mathcal{O}\})$ of the m.c.v., put $dx^j = 0$ ($j \neq i$), thus getting

$$g_{00}(\{\mathcal{O}\})c^2 dt^2 + g_{ii}(\{\mathcal{O}\})(dx^i)^2 = 0. \quad (4.21)$$

3. Evidence on the l.h.s. of (4.21) a quantity with physical dimensions $\frac{[\text{space}]}{[\text{time}]} = [\text{velocity}]$; at this point, we have two different subcases:

(I) One carries to the l.h.s. of (4.21) $\frac{dx^i}{dt}$ (which amounts to consider the 3-d Euclidean product $\cdot$), thus getting an *anisotropic* m.c.v.:

$$u^i(\{\mathcal{O}\}) \equiv \frac{dx^i}{dt} = \frac{(g_{00}(\{\mathcal{O}\}))^{1/2}}{(-g_{ii}(\{\mathcal{O}\}))^{1/2}} c \quad \forall i = 1, 2, 3. \quad (4.22)$$

(II) One carries to the l.h.s. of (4.21) $(-g_{ii}(\{\mathcal{O}\}))^{1/2} \frac{dx^i}{dt}$ (which amounts to consider the 3-d deformed product $*$ defined by $-g_{ij}(\{\mathcal{O}\}) = \delta_{ij}|g_{ii}(\{\mathcal{O}\})|$), thus getting an *isotropic* m.c.v.:

$$u^i(\{\mathcal{O}\}) \equiv (-g_{ii}(\{\mathcal{O}\}))^{1/2} \frac{dx^i}{dt} = (g_{00}(\{\mathcal{O}\}))^{1/2} c \quad \forall i = 1, 2, 3. \quad (4.23)$$

The two subcases I and II differ essentially by the different way of implementing the space anisotropy. In the former case, the anisotropy is embedded in the definition of m.c.v.; in the latter one, in the scalar product.^a

Specializing the above equations to the DSR framework, we get therefore, in the two subcases:

$$(I) \quad u_{DSR,I}^i(E) \equiv u^i(E) = c \frac{b_0(E)}{b_i(E)} \quad (4.24)$$

^aLet us notice that the directionally separating procedure can be consistently applied only to (special- or general relativistic) metrics which are fully diagonal. This is obviously due to the mixings between different space directions which arise in the case of non-diagonal metrics.

Of course, such a procedure gives (in either subcase) the same standard result when applied to SR. In fact:

$$\begin{aligned} u_{SR}^i &= (-g_{ii})^{1/2} \frac{dx^i}{dt} = (g_{00})^{1/2} c = \frac{dx^i}{dt} \\ &= \frac{(g_{00})^{1/2}}{(-g_{ii})^{1/2}} c = c \quad \forall i = 1, 2, 3. \end{aligned}$$

$$\begin{aligned}
 |\mathbf{u}_{DSR,I}(E)| &= \left(\sum_{i=1}^3 (u_{DSR,I}^i(E))^2 \right)^{1/2} \\
 &= cb_0(E) \left(\frac{1}{b_1^2(E)} + \frac{1}{b_2^2(E)} + \frac{1}{b_3^2(E)} \right)^{1/2}
 \end{aligned} \quad (4.25)$$

The vector $\mathbf{u}$ is the (spatially) anisotropic generalization of the maximal causal speed derived in the (spatially) isotropic case, Eq. (3.8);

$$(II) \quad u_{DSR,II}^i(E) \equiv w^i(E) = cb_0(E) \quad (4.26)$$

$$\begin{aligned}
 |\mathbf{u}_{DSR,II}(E)|_* &= \left(\sum_{i=1}^3 b_i^2(E) (u_{DSR,II}^i(E))^2 \right)^{1/2} \\
 &= cb_0(E) (b_1^2(E) + b_2^2(E) + b_3^2(E))^{1/2}
 \end{aligned} \quad (4.27)$$

whence

$$u_{DSR,II}^i(E) = (b_1^2(E) + b_2^2(E) + b_3^2(E))^{-1/2} |\mathbf{u}_{DSR,II}(E)|_* \quad (4.28)$$

i.e. in this subcase (unlike the previous one, see Eqs. (4.24) and (4.25)) one can state a proportionality relation by an overall factor (even if dependent on the metric coefficients) between $u_{DSR,II}^i(E)$ and $|\mathbf{u}_{DSR,II}(E)|_*$.

We have therefore shown that the two different procedures of directional separation lead to two different mathematical definitions of maximal causal velocity, an isotropic ($\mathbf{w}$, Eq. (4.26)) and an anisotropic ($\mathbf{u}$, Eq. (4.24)) one. The choice between them must be done on a physical basis (see Subsec. 4.4.3).

4.4. Boosts in DSR

4.4.1. Boost in a Generic Direction

In this case, the relative velocity is $\mathbf{v} = v^1 \hat{x} + v^2 \hat{y} + v^3 \hat{z}$, and we have to suitably generalize definitions (4.10), (4.11) as follows:

$$\tilde{\boldsymbol{\beta}} \equiv \frac{\mathbf{v}}{\mathbf{u}} \equiv \left(\frac{v^1 b_1(E)}{cb_0(E)}, \frac{v^2 b_2(E)}{cb_0(E)}, \frac{v^3 b_3(E)}{cb_0(E)} \right) \quad (4.29)$$

$$\tilde{\gamma} \equiv \left(1 - |\tilde{\boldsymbol{\beta}}|^2 \right)^{-1/2} \quad (4.30)$$

where (cf. Eq. (4.17))

$$\mathbf{u} = \left(c \frac{b_0(E)}{b_1(E)}, c \frac{b_0(E)}{b_2(E)}, c \frac{b_0(E)}{b_3(E)} \right). \quad (4.31)$$



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Notice that $\tilde{\beta} \equiv \frac{\mathbf{v}}{\mathbf{u}} \neq \frac{\mathbf{v}}{u}$. This follows from the anisotropy of the 3-vector $\mathbf{u}$, and it is to be compared with the SR case, where $\beta \equiv \frac{\mathbf{v}}{\mathbf{u}} = \frac{v}{c}$. In general, it is possible to state that

$$\frac{\mathbf{m}}{\mathbf{n}} = \frac{1}{n} \mathbf{m} \Leftrightarrow \mathbf{n} = (n, n, n)$$

i.e. iff $\mathbf{n}$ is a spatially isotropic 3-vector.

In order to derive the expression of the deformed boost in a generic direction, it is possible to use the same method of the previous case (see Appendix). However, it is simpler to consider the notion of parallelism between 3-vectors in $\tilde{M}(E)^b$ and decompose the space vector $\mathbf{x}$ in two components, $\mathbf{x}_{\parallel}$ and $\mathbf{x}_{\perp}$, parallel and orthogonal, respectively, to the boost direction $\hat{v}$

$$\mathbf{x} = \mathbf{x}_{\parallel} + \mathbf{x}_{\perp} \quad (4.32)$$

$$\begin{aligned} \mathbf{x}_{\parallel} &\equiv \hat{v}(\hat{v} * \mathbf{x}) = \frac{\mathbf{v}}{|\mathbf{v}|_*^2} (\mathbf{v} * \mathbf{x}) = \frac{\mathbf{v}}{\mathbf{v} * \mathbf{v}} (\mathbf{v} * \mathbf{x}) \\ &= \frac{\sum_{i=1}^3 b_i^2(E) v^i x^i}{\sum_{i=1}^3 b_i^2(E) (v^i)^2} \mathbf{v} \neq \hat{\beta}(\hat{\beta} * \mathbf{x}) \\ &= \frac{\tilde{\beta}}{|\tilde{\beta}|_*^2} (\tilde{\beta} * \mathbf{x}) = \frac{\tilde{\beta}}{\tilde{\beta} * \tilde{\beta}} (\tilde{\beta} * \mathbf{x}) = \frac{\sum_{i=1}^3 b_i^2(E) \tilde{\beta}^i x^i}{\sum_{i=1}^3 b_i^2(E) (\tilde{\beta}^i)^2} \tilde{\beta} \end{aligned} \quad (4.33)$$

(with $||_*$ denoting the absolute value of a 3-vector with respect to the deformed scalar product $*$, whereas the notation $||$ will be used for the 3-vector norm with respect to the standard product $\cdot$)

$$x_{\parallel}^i \equiv \frac{\sum_{k=1}^3 b_k^2(E) v^k x^k}{\sum_{k=1}^3 b_k^2(E) (v^k)^2} v^i \neq \frac{\sum_{k=1}^3 b_k^2(E) \tilde{\beta}^k x^k}{\sum_{k=1}^3 b_k^2(E) (\tilde{\beta}^k)^2} \tilde{\beta}^i \quad (4.34)$$

$$\begin{aligned} \mathbf{x}_{\perp} &\equiv \mathbf{x} - \mathbf{x}_{\parallel} = \mathbf{x} - \frac{\sum_{i=1}^3 b_i^2(E) v^i x^i}{\sum_{i=1}^3 b_i^2(E) (v^i)^2} \mathbf{v} \\ &\neq \mathbf{x} - \frac{\sum_{i=1}^3 b_i^2(E) \tilde{\beta}^i x^i}{\sum_{i=1}^3 b_i^2(E) (\tilde{\beta}^i)^2} \tilde{\beta} \end{aligned} \quad (4.35)$$

^bThe definitions of parallelism and orthogonality are to be meant in the sense of the deformed 3D scalar product $*$ (see Eq. (2.5)).

$$\begin{aligned}
x_{\perp}^i &\equiv x^i - \frac{\sum_{k=1}^3 b_k^2(E) v^k x^k}{\sum_{k=1}^3 b_k^2(E) (v^k)^2} v^i \\
&\neq x^i - \frac{\sum_{k=1}^3 b_k^2(E) \tilde{\beta}^k x^k}{\sum_{k=1}^3 b_k^2(E) (\tilde{\beta}^k)^2} \tilde{\beta}^i.
\end{aligned} \tag{4.36}$$

It is easily checked that indeed

$$\begin{aligned}
\mathbf{x} * \mathbf{v} &= \sum_{i=1}^3 b_i^2(E) x^i v^i = \frac{\sum_{i=1}^3 b_i^2(E) x^i v^i}{\sum_{i=1}^3 b_i^2(E) (v^i)^2} \sum_{k=1}^3 b_k^2(E) (v^k)^2 \\
&= \frac{\sum_{i=1}^3 b_i^2(E) x^i v^i}{\sum_{i=1}^3 b_i^2(E) (v^i)^2} \mathbf{v} * \mathbf{v} = \mathbf{x}_{\parallel} * \mathbf{v} = |\mathbf{x}_{\parallel}|_* |\mathbf{v}|_*
\end{aligned} \tag{4.37}$$

$$\mathbf{x}_{\perp} * \mathbf{v} = \mathbf{x} * \mathbf{v} - \mathbf{x}_{\parallel} * \mathbf{v} = 0. \tag{4.38}$$

Then, applying the boost (4.9) to $\mathbf{x}_{\parallel}$ and $\mathbf{x}_{\perp}$ yields

$$\begin{cases} \mathbf{x}'_{\parallel} = \tilde{\gamma}(\mathbf{x}_{\parallel} - \mathbf{v}t) \\ \mathbf{x}'_{\perp} = \mathbf{x}_{\perp} \\ t' = \tilde{\gamma} \left(t - \sum_{i=1}^3 \frac{v^i b_i^2(E)}{c^2 b_0^2(E)} x^i \right) = \tilde{\gamma}(t - \tilde{\mathbf{B}} \cdot \mathbf{x}) = \tilde{\gamma}(t - \tilde{\mathbf{B}}^{(*)} * \mathbf{x}) \end{cases} \tag{4.39}$$

where we put

$$\begin{aligned}
\tilde{\gamma} &\equiv (1 - \tilde{\beta} \cdot \tilde{\beta})^{-1/2} = (1 - \tilde{\beta}^{(*)} * \tilde{\beta}^{(*)})^{-1/2} \\
&= \left[1 - \left(\frac{v^1 b_1(E)}{c b_0(E)} \right)^2 - \left(\frac{v^2 b_2(E)}{c b_0(E)} \right)^2 - \left(\frac{v^3 b_3(E)}{c b_0(E)} \right)^2 \right]^{-1/2}
\end{aligned} \tag{4.40}$$

$$\tilde{\beta}^{(*)} \equiv \frac{\mathbf{v}}{\mathbf{w}} = \left(\frac{v^1}{c b_0(E)}, \frac{v^2}{c b_0(E)}, \frac{v^3}{c b_0(E)} \right) = \frac{1}{c b_0(E)} \mathbf{v} \tag{4.41}$$

$$\mathbf{w} \equiv (c b_0(E), c b_0(E), c b_0(E)) \tag{4.42}$$

$$\tilde{\mathbf{B}} \equiv \frac{\mathbf{v}}{\mathbf{u}^2} = \left(\frac{v^1 b_1^2(E)}{c^2 b_0^2(E)}, \frac{v^2 b_2^2(E)}{c^2 b_0^2(E)}, \frac{v^3 b_3^2(E)}{c^2 b_0^2(E)} \right) \tag{4.43}$$

$$\tilde{\mathbf{B}}^{(*)} \equiv \frac{\mathbf{v}}{\mathbf{w}^2} = \frac{1}{c^2 b_0^2(E)} \mathbf{v}. \tag{4.44}$$

It follows therefore that the deformed boosts admit a double treatment, either:

- (I) In terms of the Euclidean scalar product $\cdot$, of the (anisotropic) m.c.v. $\mathbf{u}$ and of the related velocity parameters $\tilde{\beta}$ and $\tilde{\mathbf{B}}$, or

(II) in terms of the deformed product $*$, of the (isotropic) m.c.v. $\mathbf{w}$ and of the related velocity parameters $\tilde{\beta}^{(*)}$ and $\tilde{\mathbf{B}}^{(*)c}$.

Then, the space vector transforms as:

$$\begin{aligned}\mathbf{x}' &= \mathbf{x}'_{\parallel} + \mathbf{x}'_{\perp} = \tilde{\gamma}(\mathbf{x}_{\parallel} - \mathbf{v}t) + \mathbf{x}_{\perp} \\ &= \mathbf{x} + (\tilde{\gamma} - 1)\hat{v}(\hat{v} * \mathbf{x}) - \tilde{\gamma}\mathbf{v}t = \mathbf{x} + (\tilde{\gamma} - 1)\frac{\mathbf{v}}{|\mathbf{v}|_*^2}(\mathbf{v} * \mathbf{x}) - \tilde{\gamma}\mathbf{v}t\end{aligned}\quad (4.45)$$

and we eventually find the expression of the deformed boost in a generic direction:

$$\begin{cases} \mathbf{x}' = \mathbf{x} + (\tilde{\gamma} - 1)\frac{\mathbf{v}}{|\mathbf{v}|_*^2}(\mathbf{v} * \mathbf{x}) - \tilde{\gamma}\mathbf{v}t \\ t' = \tilde{\gamma}(t - \tilde{\mathbf{B}} \cdot \mathbf{x}) = \tilde{\gamma}(t - \tilde{\mathbf{B}}^{(*)} * \mathbf{x}). \end{cases}\quad (4.46)$$

4.4.2. Symmetrization of Deformed Boosts

As in the case of standard SR, it is possible to symmetrize the expression of boosts in DSR by introducing suitable time coordinates.

Let us first consider a deformed boost along $\hat{x}^i$ ($i = 1, 2, 3$); the symmetrization transformation (a “dimensionally homogenizing dilatation”) of t is given by

$$x^0 \equiv u^i t = c \frac{b_0(E)}{b_i(E)} t; \quad x^{i'} \equiv x^i. \quad (4.47)$$

The deformed metric tensor in the new “primed” coordinates, $\{x^{\mu'}\} = \{x^0, x, y, z\}$, reads:

$$\begin{aligned}\eta'_{\mu\nu}(E) &\stackrel{\text{ESC}}{=} \eta_{\alpha\beta}(E) \frac{\partial x^\alpha}{\partial x^{\mu'}} \frac{\partial x^\beta}{\partial x^{\nu'}} = \text{diag}(b_i^2(E), -b_1^2(E), -b_2^2(E), -b_3^2(E)) \\ &\stackrel{\text{ESC}}{=} \delta_{\mu\nu} [b_i^2(E)\delta_{\mu 0} - b_1^2(E)\delta_{\mu 1} - b_2^2(E)\delta_{\mu 2} - b_3^2(E)\delta_{\mu 3}].\end{aligned}\quad (4.48)$$

Equation (4.9) takes therefore the symmetric form in x^i e x^0 (ESC off):

$$\begin{cases} x^{i'} = \tilde{\gamma}(x^i - \tilde{\beta}^i x^0); \\ x^{k \neq i'} = x^{k \neq i}; \\ x^{0'} = \tilde{\gamma}(x^0 - \tilde{\beta}^i x^i). \end{cases}\quad (4.49)$$

Transformation (4.49) does not symmetrize the deformed boost in a generic direction (unlike the case of SR, where the same transformation

^cIt is possible to show that, in this case, more equivalent forms of the deformed boost (4.39) exist. As is easily seen, this is due to the fact that, in general, $\hat{\beta} \neq \hat{v}$ and $\hat{B} \neq \hat{v}$, whereas $\tilde{\beta}^{(*)} = \hat{v} = \tilde{B}^{(*)}$.

$x^0 = ct$ symmetrizes both boosts). In this case, the symmetrization is possible only if the treatment II (based on the deformed scalar product $*$) is used.

In fact, by using the proportionality (see Eqs. (4.41) and (4.44) and the footnote at p. 36) among $\tilde{\beta}^{(*)}$, $\tilde{\mathbf{B}}^{(*)}$ and $\mathbf{v}$, the following transformation on t (see Eq. (4.42))

$$x^0 \equiv cb_0(E)t = w^k t \quad (\forall k = 1, 2, 3); \quad x^{i'} \equiv x^i \quad (\forall i = 1, 2, 3) \quad (4.50)$$

does symmetrize Eq. (4.39) in $\mathbf{x}_{\parallel}$ e x^0 :

$$\begin{cases} \mathbf{x}'_{\parallel} = (1 - \tilde{\beta}^{(*)} * \tilde{\beta}^{(*)})^{-1/2} (\mathbf{x}_{\parallel} - \tilde{\beta}^{(*)} x^0) \\ \mathbf{x}'_{\perp} = \mathbf{x}_{\perp} \\ x^{0'} = \begin{cases} (1 - \tilde{\beta}^{(*)} * \tilde{\beta}^{(*)})^{-1/2} (x^0 - \tilde{\beta}^{(*)} * \mathbf{x}) \\ = (1 - \tilde{\beta}^{(*)} * \tilde{\beta}^{(*)})^{-1/2} (x^0 - \tilde{\beta}^{(*)} * \mathbf{x}_{\parallel}). \end{cases} \end{cases} \quad (4.51)$$

Under transformation (4.51), the metric tensor becomes:

$$\eta'_{\mu\nu}(E) \stackrel{\text{ESC}}{=} \eta_{\alpha\beta}(E) \frac{\partial x^{\alpha}}{\partial x^{\mu'}} \frac{\partial x^{\beta}}{\partial x^{\nu'}} = \text{diag}(1, -b_1^2(E), -b_2^2(E), -b_3^2(E)) \stackrel{\text{ESC}}{=} \delta_{\mu\nu} [\delta_{\mu 0} - b_1^2(E) \delta_{\mu 1} - b_2^2(E) \delta_{\mu 2} - b_3^2(E) \delta_{\mu 3}]. \quad (4.52)$$

Therefore the symmetrization of the deformed boost in a generic direction makes the 4-d metric isochronous, since $\eta'_{00} = 1$ so that $\tau = t$ (namely proper time coincides with coordinate time).

Let us finally notice that, like in the SR case, the boost in generic direction expressed in terms of $\mathbf{x}$ e t (Eq. (4.39)) cannot in general be symmetrized.

Apparently Eqs. (4.9) are asymmetrical in the behaviour of x' and t' , unlike the usual Lorentz transformations, which are fully symmetric when putting $x^0 = ct$. However, such asymmetry is only formal. It can be removed by introducing, in analogy with the electromagnetic case, a time coordinate defined in terms of the maximal causal speed u in the generalized Minkowski space considered:

$$x^0 = ut = \left(\frac{b_0}{b} c \right) t \quad (4.53)$$

and changing the metric tensor η into

$$\eta' = \text{diag}(b^2, -b^2, -b^2, -b^2) = b^2 g. \quad (4.54)$$



Then, the generalized Lorentz transformations in $\tilde{M}'$ take the symmetrical form

$$\begin{cases} x^{0'} = \tilde{\gamma}(x^0 - \tilde{\beta}x^1); \\ x^{1'} = \tilde{\gamma}(x^1 - \tilde{\beta}x^0); \\ x^{2'} = x^2; \\ x^{3'} = x^3. \end{cases} \quad (4.55)$$

It is easily seen that the deformed Minkowski spaces $\tilde{M}$ and $\tilde{M}'$, with metrics (2.2) and (4.54) respectively, are isometric, because they have the same interval (2.5). They are therefore fully equivalent in every respect, and it is therefore possible to use indifferently either transformation (4.9) or (4.55). The main advantage of the latter ones is that, due to relation (4.53), the formulae holding for $\tilde{M}'$ are immediately got from those of the standard special relativity by simply replacing everywhere c by u .

4.4.3. Velocity Composition Law in $\tilde{M}$ and the Invariant Maximal Speed

We have seen in Sec. 4.3 that the directionally separating approach (mandatory in the deformed case) yields two different *mathematical* definitions $\mathbf{u}$ (Eq. (4.24)) and $\mathbf{w}$ (Eq. (4.26)) of maximal causal velocity in DSR. The choice between them must be done on a physical basis, by checking their actual invariance under deformed boosts.

To this aim, one has to derive the generalized velocity composition law valid in $\tilde{M}$. For a deformed boost in the direction $\hat{x}^i$, differentiating the inverse of Eq. (4.9) yields (on account of the fact that $dE = 0$ in DSR) (ESC off):

$$\begin{cases} dx^i = \tilde{\gamma}(dx^{i'} + v^i dt') \\ dx^{k \neq i} = dx^{k \neq i'} \\ dt = \tilde{\gamma} \left(dt' + \frac{v^i b_i^2(E)}{c^2 b_0^2(E)} dx^{i'} \right) \end{cases} \quad (4.56)$$

with $\tilde{\gamma}$ given by (4.11). Since

$$\frac{dx^i}{dt} = v^i, \quad \frac{dx^{i'}}{dt'} = v^{i'}, \quad \frac{dx^{k \neq i}}{dt} = v^{k \neq i}, \quad \frac{dx^{k \neq i'}}{dt'} = v^{k \neq i'} \quad (4.57)$$

one gets the *deformed velocity composition law* (in compact notation, ESC off)

$$v^k = \frac{v^{k'} + \delta_{ik} v^i}{\left[1 + \left(\frac{b_i(E)}{b_0(E)} \right)^2 \frac{v^i v^{i'}}{c^2} \right] \{ \tilde{\gamma}(E) + \delta_{ik} [1 - \tilde{\gamma}(E)] \}}. \quad (4.58)$$

This relation can be expressed in terms of the standard 3-d scalar product $\cdot$ (and therefore of the anisotropic maximal velocity $\mathbf{u}$) (approach I) as

$$\begin{aligned} v^k &= \frac{v^{k'} + \delta_{ik} v^i}{\left[1 + \frac{\mathbf{v} \cdot \mathbf{v}'}{(u^i(E))^2} \right] \{ \tilde{\gamma}(E) + \delta_{ik} [1 - \tilde{\gamma}(E)] \}} \\ &= \frac{v^{k'} + \delta_{ik} v^i}{\left[1 + \frac{\tilde{\beta} \cdot \mathbf{v}'}{u^i(E)} \right] \{ \tilde{\gamma}(E) + \delta_{ik} [1 - \tilde{\gamma}(E)] \}} \end{aligned} \quad (4.59)$$

where

$$\tilde{\beta}^i(E) = \frac{v^i}{u^i(E)}; \quad \tilde{\gamma}(E) = (1 - \tilde{\beta}(E) \cdot \tilde{\beta}(E))^{-1/2}. \quad (4.60)$$

Alternatively, we can use approach II, based on the deformed scalar product $*$ (and therefore the isotropic maximal velocity $\mathbf{w}$) and write Eq. (4.59) as

$$\begin{aligned} v^k &= \frac{v^{k'} + \delta_{ik} v^i}{\left[1 + \frac{\mathbf{v} * \mathbf{v}'}{(w^i(E))^2} \right] \{ \tilde{\gamma}(E) + \delta_{ik} [1 - \tilde{\gamma}(E)] \}} \\ &= \frac{v^{k'} + \delta_{ik} v^i}{\left[1 + \frac{\tilde{\beta}^{(*)} * \mathbf{v}'}{w^i(E)} \right] \{ \tilde{\gamma}(E) + \delta_{ik} [1 - \tilde{\gamma}(E)] \}} \end{aligned} \quad (4.61)$$

with

$$\tilde{\beta}^{(*)i}(E) = \frac{v^i}{w^i(E)}; \quad \tilde{\gamma}(E) = (1 - \tilde{\beta}^{(*)}(E) * \tilde{\beta}^{(*)}(E))^{-1/2}. \quad (4.62)$$

It is now an easy task to check the truly maximal character of the two velocities. Indeed, if $v^{i'} = u^i(E)$, one gets, from Eq. (4.59)

$$v^i = \frac{u^i(E) + v^i}{1 + \frac{v^i}{u^i(E)}} = u^i(E) \quad (4.63)$$

whereas, for $v^i = w^i(E)$, Eq. (4.61) yields

$$v^i = \frac{w^i(E) + v^i}{1 + \frac{(b_i(E))^2 v^i}{w^i(E)}} \neq w^i(E). \quad (4.64)$$

We can therefore conclude, on a physical basis, that $\mathbf{u}$ is the maximal, invariant causal velocity in DSR, and it plays in the deformed Minkowski space $\tilde{M}$ the role of the light speed in standard SR.^d

It is also easy to see why — although approach (II) looks at first sight more rigorous mathematically, because it permits to connect the peculiar features of spatial anisotropy of DSR to the deformed product $*$, “naturally induced” from the metric of $\tilde{M}(E)$ — actually it’s approach (I) which yields the physically relevant result. Indeed, the velocity $\mathbf{u}$ is just defined as $\frac{d\mathbf{x}}{dt}$, and it therefore represents the physically measured velocity, for a particle moving in the usual, physical Euclidean 3-d space. On the other hand, this result clearly shows that the space anisotropy introduced by the deformed metric is not a mere mathematical artifact, but it reflects itself in the physical properties (imposed by the interaction involved) of the phenomenon described by the deformed space-time.

The comparison of the deformed boost expression (Eq. (4.9)) with the corresponding ones of the standard Lorentz boosts shows clearly that the transition from SR (based on M) to DSR (based on $\tilde{M}$) is simply carried out by letting

$$\mathbf{u}_{SR} = (c, c, c) \rightarrow \mathbf{u}_{DSR}(E) = \left(\frac{cb_0(E)}{b_1(E)}, \frac{cb_0(E)}{b_2(E)}, \frac{cb_0(E)}{b_3(E)} \right). \quad (4.65)$$

In other words, the difference between M and $\tilde{M}(E)$ (at least as far as the finite coordinate transformations are concerned) is completely embodied in the 3-vector m.c.v. $\mathbf{u}$.

^dOf course, in the case of space isotropy, we get an isotropic maximal causal velocity given by (cf. Eq. (2.9))

$$u_{\text{iso}}^i(E) = u_{DSR,II}^i(E)|_{b_i(E)=b(E)} = c \frac{b_0(E)}{b(E)} \quad \forall i = 1, 2, 3$$

$$|\mathbf{u}_{\text{iso}}(E)| = \left(\sum_{i=1}^3 (u_{\text{iso}}^i(E))^2 \right)^{1/2} = \sqrt{3}c \frac{b_0(E)}{b(E)}$$