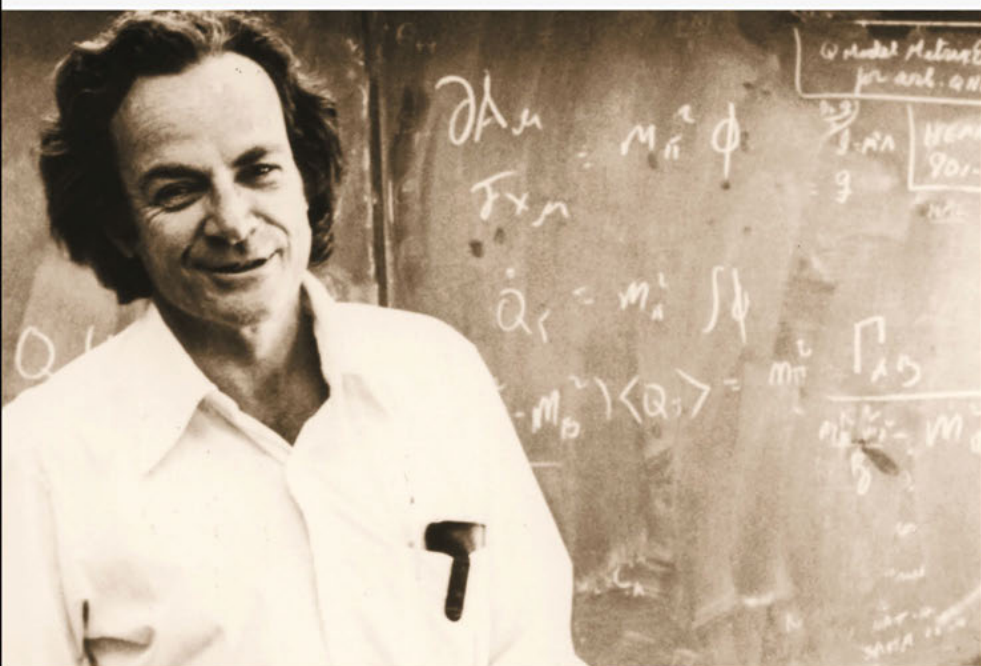


FEYNMAN'S THESIS



A New Approach to Quantum Theory

Laurie M. Brown (Editor)

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A New Approach to Quantum Theory

Editor

Laurie M. Brown

Northwestern University, USA



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Published by

World Scientific Publishing Co. Pte. Ltd.

5 Toh Tuck Link, Singapore 596224

USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601

UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

Cover image: AIP Emilio Segrè Visual Archives, Physics Today Collection

THE PRINCIPLE OF LEAST ACTION IN QUANTUM MECHANICS

by Richard P. Feynman is published by arrangement through Big Apple Tuttle-Mori Agency.

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FEYNMAN'S THESIS — A NEW APPROACH TO QUANTUM THEORY

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ISBN 981-256-366-0

ISBN 981-256-380-6 (pbk)

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Contents

Preface	vii
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The Principle of Least Action in Quantum Mechanics

R. P. Feynman

I. Introduction	1
II. Least Action in Classical Mechanics	6
1. The Concept of Functional	6
2. The Principle of Least Action	9
3. Conservation of Energy. Constants of the Motion	10
4. Particles Interacting through an Intermediate Oscillator	16
III. Least Action in Quantum Mechanics	24
1. The Lagrangian in Quantum Mechanics	26
2. The Calculation of Matrix Elements in the Language of a Lagrangian	32
3. The Equations of Motion in Lagrangian Form	34
4. Translation to the Ordinary Notation of Quantum Mechanics	39
5. The Generalization to Any Action Function	41
6. Conservation of Energy. Constants of the Motion	42
7. The Role of the Wave Function	44
8. Transition Probabilities	46
9. Expectation Values for Observables	49
10. Application to the Forced Harmonic Oscillator	55
11. Particles Interacting through an Intermediate Oscillator	61
12. Conclusion	68

Space-time Approach to Non-Relativistic Quantum Mechanics	71
<i>R. P. Feynman</i>	
The Lagrangian in Quantum Mechanics	111
<i>P. A. M. Dirac</i>	

Preface

Since Richard Feynman's death in 1988 it has become increasingly evident that he was one of the most brilliant and original theoretical physicists of the twentieth century.¹ The Nobel Prize in Physics for 1965, shared with Julian Schwinger and Sin-itiro Tomonaga, rewarded their independent path-breaking work on the renormalization theory of quantum electrodynamics (QED). Feynman based his own formulation of a consistent QED, free of meaningless infinities, upon the work in his doctoral thesis of 1942 at Princeton University, which is published here for the first time. His new approach to quantum theory made use of the Principle of Least Action and led to methods for the very accurate calculation of quantum electromagnetic processes, as amply confirmed by experiment. These methods rely on the famous "Feynman diagrams," derived originally from the path integrals, which fill the pages of many articles and textbooks. Applied first to QED, the diagrams and the renormalization procedure based upon them also play a major role in other quantum field theories, including quantum gravity and the current "Standard Model" of elementary particle physics. The latter theory involves quarks and leptons interacting through the exchange of renormalizable Yang–Mills non-Abelian gauge fields (the electroweak and color gluon fields).

The path-integral and diagrammatic methods of Feynman are important general techniques of mathematical physics that have many applications other than quantum field theories: atomic and molecular scattering, condensed matter physics, statistical mechanics, quantum liquids and solids, Brownian motion, noise, etc.² In addition to

¹ Hans Bethe's obituary of Feynman [*Nature* **332** (1988), p. 588] begins: "Richard P. Feynman was one of the greatest physicists since the Second World War and, I believe, the most original."

² Some of these topics are treated in R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, Massachusetts, 1965). Also see M. C. Gutzwiller, "Resource Letter ICQM-1: The Interplay Between Classical and Quantum Mechanics," *Am. J. Phys.* **66** (1998), pp. 304–24; items 71–73 and **Documents** path integrals.

its usefulness in these diverse fields of physics, the path-integral approach brings a new fundamental understanding of quantum theory. Dirac, in his transformation theory, demonstrated the complementarity of two seemingly different formulations: the matrix mechanics of Heisenberg, Born, and Jordan and the wave mechanics of de Broglie and Schrödinger. Feynman's independent path-integral theory sheds new light on Dirac's operators and Schrödinger's wave functions, and inspires some novel approaches to the still somewhat mysterious interpretation of quantum theory. Feynman liked to emphasize the value of approaching old problems in a new way, even if there were to be no immediate practical benefit.

Early Ideas on Electromagnetic Fields

Growing up and educated in New York City, where he was born on 11 May 1918, Feynman did his undergraduate studies at the Massachusetts Institute of Technology (MIT), graduating in 1939. Although an exceptional student with recognized mathematical prowess, he was not a prodigy like Julian Schwinger, his fellow New Yorker born the same year, who received his PhD in Physics from Columbia University in 1939 and had already published fifteen articles. Feynman had two publications at MIT, including his undergraduate thesis with John C. Slater on "Forces and Stresses in Molecules." In that work he proved a very important theorem in molecular and solid-state physics, which is now known as the Hellmann–Feynman theorem.³

While still an undergraduate at MIT, as he related in his Nobel address, Feynman devoted much thought to electromagnetic interactions, especially the self-interaction of a charge with its own field, which predicted that a pointlike electron would have an infinite mass. This unfortunate result could be avoided in classical physics, either by not calculating the mass, or by giving the theoretical electron an

³ L. M. Brown (ed.), *Selected Papers of Richard Feynman, with Commentary* (World Scientific, Singapore, 2000), p. 3. This volume (hereafter referred to as *SP*) includes a complete bibliography of Feynman's work.

extended structure; the latter choice makes for some difficulties in relativistic physics.

Neither of these solutions are possible in QED, however, because the extended electron gives rise to non-local interaction and the infinite pointlike mass inevitably contaminates other effects, such as atomic energy level differences, when calculated to high accuracy. While at MIT, Feynman thought that he had found a simple solution to this problem: Why not assume that the electron does not experience any interaction with its own electromagnetic field? When he began his graduate study at Princeton University, he carried this idea with him. He explained why in his Nobel Address:⁴

Well, it seemed to me quite evident that the idea that a particle acts on itself is not a necessary one — it is a sort of silly one, as a matter of fact. And so I suggested to myself that electrons cannot act on themselves; they can only act on other electrons. That means there is no field at all. There was a direct interaction between charges, albeit with a delay.

A new classical electromagnetic field theory of that type would avoid such difficulties as the infinite self-energy of the point electron. The very useful notion of a field could be retained as an auxiliary concept, even if not thought to be a fundamental one. There was a chance also that if the new theory were quantized, it might eliminate the fatal problems of the then current QED. However, Feynman soon learned that there was a great obstacle to this delayed action-at-a-distance theory: namely, if a radiating electron, say in an atom or an antenna, were not acted upon at all by the field that it radiated, then it would not recoil, which would violate the conservation of energy. For that reason, some form of radiative reaction is necessary.

⁴ SP, pp. 9–32, especially p. 10.

The Wheeler-Feynman Theory

Trying to work through this problem at Princeton, Feynman asked his future thesis adviser, the young Assistant Professor John Wheeler, for help. In particular, he asked whether it was possible to consider that two charges interact in such a way that the second charge, accelerated by absorbing the radiation emitted by the first charge, itself emits radiation that reacts upon the first. Wheeler pointed out that there would be such an effect but, delayed by the time required for light to pass between the two particles, it could not be the force of radiation reaction, which is instantaneous; also the force would be much too weak. What Feynman had suggested was not radiation reaction, but the reflection of light!

However, Wheeler did offer a possible way out of the difficulty. First, one could assume that radiation always takes place in a totally absorbing universe, like a room with the blinds drawn. Second, although the principle of causality states that all observable effects take place at a time later than the cause, Maxwell's equations for the electromagnetic field possess a radiative solution other than that normally adopted, which is delayed in time by the finite velocity of light. In addition, there is a solution whose effects are advanced in time by the same amount. A linear combination of retarded and advanced solutions can also be used, and Wheeler asked Feynman to investigate whether some suitable combination in an absorbing universe would provide the required observed instantaneous radiative reaction?

Feynman worked out Wheeler's suggestion and found that, indeed, a mixture of one-half advanced and one-half retarded interaction in an absorbing universe would exactly mimic the result of a radiative reaction due to the electron's own field emitting purely retarded radiation. The advanced part of the interaction would stimulate a response in the electrons of the absorber, and their effect at the source (summed over the whole absorber) would arrive at just the right time and in the right strength to give the required radiation reaction force, without assuming any direct interaction of the electron with its own radiation field. Furthermore, no apparent

violation of the principle of causality arises from the use of advanced radiation. Wheeler and Feynman further explored this beautiful theory in articles published in the *Reviews of Modern Physics* (RMP) in 1945 and 1949.⁵ In the first of these articles, no less than four different proofs are presented of the important result concerning the radiative reaction.

Quantizing the Wheeler–Feynman theory (Feynman’s PhD thesis): *The Principle of Least Action in Quantum Mechanics*

Having an action-at-a-distance classical theory of electromagnetic interactions without fields, except as an auxiliary device, the question arises as to how to make a corresponding quantum theory. To treat a classical system of interacting particles, there are available analytic methods using generalized coordinates, developed by Hamilton and Lagrange, corresponding canonical transformations, and the principle of least action.⁶ The original forms of quantum mechanics, due to Heisenberg, Schrödinger, and Dirac, made use of the Hamiltonian approach and its consequences, especially Poisson brackets. To quantize the electromagnetic field it was represented, by Fourier transformation, as a superposition of plane waves having transverse, longitudinal, and timelike polarizations. A given field was represented as mathematically equivalent to a collection of harmonic oscillators. A system of interacting particles was then described by a Hamiltonian function of three terms representing respectively the particles, the field, and their interaction. Quantization consisted of regarding these terms as Hamiltonian *operators*, the field’s Hamiltonian describing a suitable infinite set of quantized harmonic oscillators. The combination of longitudinal and timelike oscillators

⁵ *SP*, p. 35–59 and p. 60–68. The second paper was actually written by Wheeler, based upon the joint work of both authors. It is remarked in these papers that H. Tetrode, W. Ritz, and G. N. Lewis had independently anticipated the absorber idea.

⁶ W. Yourgrau and S. Mandelstam give an excellent analytic historical account in *Variational Principles in Dynamics and Quantum Theory* (Saunders, Philadelphia, 3rd edn., 1968).

was shown to provide the (instantaneous) Coulomb interaction of the particles, while the transverse oscillators were equivalent to photons. This approach, as well as the more general approach adopted by Heisenberg and Pauli (1929), was based upon Bohr's correspondence principle.

However, no method based upon the Hamiltonian could be used for the Wheeler–Feynman theory, either classically or quantum mechanically. The principal reason was the use of half-advanced and half-retarded interaction. The Hamiltonian method describes and keeps track of the state of the system of particles and fields at a given time. In the new theory, there are no field variables, and every radiative process depends on contributions from the future as well as from the past! One is forced to view the entire process from start to finish. The only existing classical approach of this kind for particles makes use of the principle of least action, and Feynman's thesis project was to develop and generalize this approach so that it could be used to formulate the Wheeler–Feynman theory (a theory possessing an action, but without a Hamiltonian). If successful, he should then try to find a method to quantize the new theory.⁷

The Introduction to the Thesis

Presenting his motivation and giving the plan of the thesis, Feynman's introductory section laid out the principal features of the (not yet published) delayed electromagnetic action-at-a-distance theory as described above, including the postulate that “fundamental (microscopic) phenomena in nature are symmetrical with respect to the interchange of past and future.” Feynman claimed: “This requires that the solution of Maxwell's equation[s] to be used in computing the interactions is to be half the retarded plus half the advanced solution of Lienard and Wiechert.” Although it would appear to contradict causality, Feynman stated that the principles of

⁷ For a related discussion, including Feynman's PhD thesis, see S. S. Schweber, *QED and the Men Who Made It: Dyson, Feynman, Schwinger, and Tomonaga* (Princeton University Press, Princeton, 1994), especially pp. 389–397.

the theory “do in fact lead to essential agreement with the results of the more usual form of electrodynamics, and at the same time permit a consistent description of point charges and lead to a unique law of radiative damping It is shown that these principles are equivalent to the equations of motion resulting from a principle of least action.”

To explain the spontaneous decay of excited atoms and the existence of photons, both seemingly contradicted by this view, Feynman argued that “an atom alone in empty space would, in fact, *not* radiate . . . and all of the apparent quantum properties of light and the existence of photons may be nothing more than the result of matter interacting with matter directly, and according to quantum mechanical laws.”

Two important points conclude the introduction. First, although the Wheeler–Feynman theory clearly furnished its motivation: “It is to be emphasized . . . that the work described here is complete in itself without regard to its application to electrodynamics . . . [The] present paper is concerned with the problem of finding a quantum mechanical description applicable to systems which in their classical analogue are expressible by a principle of least action, and not necessarily by Hamiltonian equations of motion.” The second point is this: “All of the analysis will apply to non-relativistic systems. The generalization to the relativistic case is not at present known.”

Classical Dynamics Generalized

The second section of the thesis discusses the theory of functionals and functional derivatives, and it generalizes the principle of least action of classical dynamics. Applying this method to the particular example of particles interacting through the intermediary of classical harmonic oscillators (an analogue of the electromagnetic field), Feynman shows how the coordinates of the oscillators can be eliminated and how their role in the interaction is replaced by a direct delayed interaction of the particles. Before this elimination process, the system consisting of oscillators and particles possesses a Hamiltonian but afterward, when the particles have direct interaction, no

Hamiltonian formulation is possible. Nevertheless, the equations of motion can still be derived from the principle of least action. This demonstration sets the stage for a similar procedure to be carried out in the quantized theory developed in the third and final section of the thesis.

In classical dynamics, the action is given by

$$S = \int L(q(t), \dot{q}(t)) dt,$$

where L is a function of the generalized coordinates $q(t)$ and the generalized velocities $\dot{q} = dq/dt$, the integral being taken between the initial and final times t_0 and t_1 , for which the set of q 's have assigned values. The action depends on the paths $q(t)$ taken by the particles, and thus it is a functional of those paths. The *principle of least action* states that for “small” variations of the paths, the end points being fixed, the action S is an extremum, in most cases a minimum. An equivalent statement is that the functional derivative of S is zero. In the usual treatment, this principle leads to the Lagrangian and Hamiltonian equations of motion.

Feynman illustrates how this principle can be extended to the case of a particle (perhaps an atom) interacting with itself through advanced and retarded waves, by means of a mirror. An interaction term of the form $k^2 \dot{x}(t) \dot{x}(t + T)$ is added to the Lagrangian of the particle in the action integral, T being the time for light to reach the mirror and return to the particle. (As an approximation, the limits of integration of the action integral are taken as negative and positive infinity.) A simple calculation, setting the variation of the action equal to zero, leads to the equation of motion of the particle. This shows that the force on the particle at time t depends on the particle's motion at times t , $t - T$, and $t + T$. That leads Feynman to observe: “The equations of motion cannot be described directly in Hamiltonian form.”

After this simple example, there is a section discussing the restrictions that are needed to guarantee the existence of the usual constants of motion, including the energy. The thesis then treats the more complicated case of particles interacting via intermediate

oscillators. It is shown how to eliminate the oscillators and obtain direct delayed action-at-a-distance. Interestingly, by making a suitable choice of the action functional, one can obtain particles either with or without self-interaction.

While still working on formulating the classical Wheeler–Feynman theory, Feynman was already beginning to adopt the over-all space-time approach that characterizes the quantization carried out in the thesis and in so much of his subsequent work, as he explained in his Nobel Lecture:⁸

By this time I was becoming used to a physical point of view different from the more customary point of view. In the customary view, things are discussed as a function of time in very great detail. For example, you have the field at this moment, a different equation gives you the field at a later moment and so on; a method, which I shall call the Hamiltonian method, a time differential method. We have, instead [the action] a thing that describes the character of the path throughout all of space and time. The behavior of nature is determined by saying her whole space-time path has a certain character. For the action [with advanced and retarded terms] the equations are no longer at all easy to get back into Hamiltonian form. If you wish to use as variables only the coordinates of particles, then you can talk about the property of the paths — but the path of one particle at a given time is affected by the path of another at a different time Therefore, you need a lot of bookkeeping variables to keep track of what the particle did in the past. These are called field variables From the overall space-time point of view of the least action principle, the field disappears as nothing but bookkeeping variables insisted on by the Hamiltonian method.

Of the many significant contribution to theoretical physics that Feynman made throughout his career, perhaps none will turn out to

⁸ “The development of the space-time view of quantum electrodynamics,” *SP*, pp. 9–32; especially p. 16.

be of more lasting value than his reformulation of quantum mechanics, complementing those of Heisenberg, Schrödinger, and Dirac.⁹ When extended to the relativistic domain and including the quantized electromagnetic field, it forms the basis of Feynman's version of QED, which is now the version of choice of theoretical physics, and which was seminal in the development of the gauge theories employed in the Standard Model of particle physics.¹⁰

Quantum Mechanics and the Principle of Least Action

The third and final section of the thesis, together with the RMP article of 1949, presents the new form of quantum mechanics.¹¹ In reply to a request for a copy of the thesis, Feynman said he had not an available copy, but instead sent a reprint of the RMP article, with this explanation of the difference:¹²

This article contains most of what was in the thesis. The thesis contained in addition a discussion of the relation between constants of motion such as energy and momentum and invariance properties of an action functional. Further there is a much more thorough discussion of the possible gen-

⁹ The action principle approach was later adopted also by Julian Schwinger. In discussing these formulations, Yourgrau and Mandelstam comment: "One cannot fail to observe that Feynman's principle in particular — and this is no hyperbole — expresses the laws of quantum mechanics in an exemplary neat and elegant manner, notwithstanding the fact that it employs somewhat unconventional mathematics. It can easily be related to Schwinger's principle, which utilizes mathematics of a more familiar nature. The theorem of Schwinger is, as it were, simply a translation of that of Feynman into differential notation." (Taken from Yourgrau and Mandelstam's book [footnote 6], p. 128.)

¹⁰ Although it had initially motivated his approach to QED, Feynman found later that the quantized version of the Wheeler–Feynman theory (that is, QED without fields) could not account for the experimentally observed phenomenon known as vacuum polarization. Thus in a letter to Wheeler (on May 4, 1951) Feynman wrote: "I wish to deny the correctness of the assumption that electrons act only on other electrons So I think we guessed wrong in 1941. Do you agree?"

¹¹ R. P. Feynman, "Space-time approach to non-relativistic quantum mechanics," *Rev. Mod. Phys.* **20** (1948) pp. 367–387 included here as an appendix. Also in *SP*, pp. 177–197.

¹² Letter to J. G. Valatin, May 11, 1949.



eralization of quantum mechanics to apply to more general functionals than appears in the Review article. Finally the properties of a system interacting through intermediate harmonic oscillators is discussed in more detail.

The introductory part of this third section of the thesis refers to Dirac's classical treatise for the usual formulation of quantum mechanics.¹³

However, Feynman writes that for those classical systems, which have no Hamiltonian form “no satisfactory method of quantization has been given.” Thus he intends to provide one, based on the principle of least action. He will show that this method satisfies two necessary criteria: First, in the limit that $\hbar$ approaches zero, the quantum mechanical equations derived approach the classical ones, including the extended ones considered earlier. Second, for a system whose classical analogue does possess a Hamiltonian, the results are completely equivalent to the usual quantum mechanics.

The next section, “The Lagrangian in Quantum Mechanics” has the same title as an article of Dirac, published in 1933.¹⁴ Dirac presents there an alternative version to a quantum mechanics based on the classical Hamiltonian, which is a function of the coordinates q and the momenta p of the system. He remarks that the Lagrangian, a function of coordinates and velocities, is more fundamental because the action defined by it is a relativistic invariant, and also because it admits a principle of least action. Furthermore, it is “closely connected to the theory of contact transformations,” which has an important quantum mechanical analogue, namely, the transformation matrix $(q_t|q_T)$. This matrix connects a representation with the variables q diagonal at time T with a representation having the q 's diagonal at time t . In the article, Dirac writes that $(q_t|q_T)$

¹³ P. A. M. Dirac, *The Principles of Quantum Mechanics* (Oxford University Press, Oxford, 2nd edn., 1935). Later editions contain very similar material regarding the fundamental aspects to which Feynman refers.

¹⁴ P. A. M. Dirac, in *Physikalische Zeitschrift der Sowjetunion*, Band 3, Heft 1 (1933), included here as an appendix. In discussing this material, Feynman includes a lengthy quotation from Dirac's *Principles*, 2nd edn., pp. 124–126.

“corresponds to” the quantity $A(tT)$, defined as

$$A(tT) = \exp \left[i \int_T^t L dt / \hbar \right] .$$

A bit later on, he writes that $A(tT)$ “is the classical analogue of $(q_t|q_T)$.”

When Herbert Jehle, who was visiting Princeton in 1941, called Feynman's attention to Dirac's article, he realized at once that it gave a necessary clue, based upon the principle of least action that he could use to quantize classical systems that do not possess a Hamiltonian. Dirac's paper argues that the classical limit condition for $\hbar$ approaching zero is satisfied, and Feynman shows this explicitly in his thesis. The procedure is to divide the time interval $t - T$ into a large number of small elements and consider a succession of transformations from one time to the next:

$$(q_t|q_T) = \iint \cdots \int (q_t|q_m) dq_m (q_m|q_{m-1}) dq_{m-1} \cdots (q_2|q_1) dq_1 (q_1|q_T) .$$

If the transformation function has a form like $A(tT)$, then the integrand is a rapidly oscillating function when $\hbar$ is small, and only those paths $(q_T, q_1, q_2, \dots, q_t)$ give an appreciable contribution for which the phase of the exponential is stationary. In the limit, only those paths are allowed for which the action is a minimum; i.e., for which $\delta S = 0$, with

$$S = \int_T^t L dt .$$

For a very small time interval ε , the transformation function takes the form

$$A(t, t + \varepsilon) = \exp iL\varepsilon/\hbar ,$$

where $L = L((Q - q)/\varepsilon, Q)$, and we have let $q = q_t$ and $Q = q_{t+\varepsilon}$. Applying the transformation function to the wave function $\psi(q, t)$ to obtain $\psi(Q, t + \varepsilon)$ and expanding the resulting integral equation to first order in ε , Feynman obtains the Schrödinger equation. His derivation is valid for any Lagrangian containing at most quadratic terms in the velocities. In this way he demonstrates two important

points: In the first place, the derivation shows that the usual results of quantum mechanics are obtained for systems possessing a classical Lagrangian from which a Hamiltonian can be derived. Second, he shows that Dirac's $A(tT)$ is not merely an analogue of $\langle q_t | q_T \rangle$, but is *equal* to it, for a small time ε , up to a normalization factor. For a single coordinate, this factor is $N = \sqrt{2\pi i \varepsilon \hbar / m}$.

This method turns out to be an extraordinarily powerful way to obtain Feynman's path-integral formulation of quantum mechanics, upon which much of his subsequent thinking and production was based. Successive application of infinitesimal transformations provides a transformation of the wave function over a finite time interval, say from time T to time t . The Lagrangian in the exponent can be approximated to first order in ε , and

$$\psi(Q, T) \cong \iint \cdots \int \exp \left\{ \frac{i}{\hbar} \sum_{i=0}^m \left[L \left(\frac{q_{i+1} - q_i}{t_{i+1} - t_i}, q_{i+1} \right) (t_{i+1} - t_i) \right] \right\} \\ \times \psi(q_0, t_0) \frac{\sqrt{g_0} dq_0 \cdots \sqrt{g_m} dq_m}{N(t_1 - t_0) \cdots N(T - t_m)},$$

is the result obtained by induction, where $Q = q_{m+1}, T = t_{m+1}$, and the N 's are the normalization factors (one for each q) referred to above. In the limit where ε goes to zero, the right-hand side is equal to $\psi(Q, T)$. Feynman writes: "The sum in the exponential resembles $\int_{t_0}^T L(q, \dot{q}) dt$ with the integral written as a Riemann sum. In a similar manner we can compute $\psi(q_0, t_0)$ in terms of the wave function at a later time ..."

A sequence of q 's for each t_i will, in the limit, define a path of the system and each of the integrals is to be taken over the entire range available to each q_i . In other words, the multiple integral is taken over all possible paths. We note that each path is continuous but not, in general, differentiable.

Using the idea of path integrals as in the expression above for $\psi(Q, T)$, Feynman considers expressions at a given time t_0 , such as $\langle f(q_0) \rangle = \langle \chi | f(q_0) | \Psi \rangle$, which represents a quantum mechanical matrix element if χ and Ψ are different state functions or an expectation value if they represent the same state (i.e., $\chi = \Psi^*$). Path

integrals relate the wave function $\psi(q_0, t_0)$ to an earlier time and the wave function $\chi(q_0, t_0)$ to a later time, which are taken as the distant past and future, respectively. By writing $\langle f(q_0) \rangle$ at two times separated by ε and letting ε approach zero, Feynman shows how to calculate the time derivative of $\langle f(q_t) \rangle$.

The next section of the thesis uses the language of functionals $F(q_i)$, depending on the values of the q 's at the sequence of times t_i , to derive the quantum Lagrangian equations of motion from the path integrals. It shows the relation of these equations to q -number equations, such as $pq - qp = \hbar/i$ and discusses the relation of the Lagrangian formulation to the Hamiltonian one for cases where the latter exists. For example, the well-known result is derived that $HF - FH = (\hbar/i)\dot{F}$.

As was the case in the discussion of the classical theory, Feynman extends the formalism to the case of a more general action functional, beginning with the simple example of "a particle in a potential $V(x)$ and which also interacts with itself in a mirror, with half advanced and half retarded waves." An immediate difficulty is that the corresponding Lagrangian function involves two times. As a consequence, the action integral over the finite interval between times T_1 and T_2 is meaningless, because "the action might depend on values of $x(t)$ outside of this range." One can avoid this difficulty by formally letting the interaction vanish at times after large positive T_2 and before large negative T_1 . Then for times outside the range of integration the particles are effectively free, so that wave functions can be defined at the endpoints. With this assumption the earlier discussions concerning functionals, operators, etc., can be carried through with the more general action functional.

However, the question as to whether a wave function or other wave-function-like object exists with the generalized Lagrangian is not solved in the thesis (and perhaps has never been solved). Although Feynman shows that much of quantum mechanics can be solved in terms of expectation values and transition amplitudes, at the end it is far from clear that it is possible to drop the very useful notion of the wave function (and if it is possible, it is probably not desirable to do so). A number of the pages of the thesis that follow

are concerned with the question of the wave function, with conservation of energy, and with the calculation of transition probability amplitudes, including the development of a perturbation theory.

We shall not discuss these issues here, but continue to the last part of the thesis, where the forced harmonic oscillator is calculated. Based upon the path-integral solution of that problem, particles interacting through an intermediate oscillator are introduced and eventually the oscillators (i.e. the “field variables”) are completely eliminated. Enrico Fermi had introduced the method of representing the electromagnetic field as a collection of oscillators and had eliminated the oscillators of longitudinal and timelike polarization to give the instantaneous Coulomb potential, as Feynman points out.¹⁵ That had been the original aim of the thesis, to eliminate *all of the oscillators* (and hence the field) in order to quantize the Wheeler–Feynman action-at-a-distance theory. It turns out, however, that the elimination of all the oscillators was also very valuable in field theory having purely retarded interaction, and led in fact to the overall space-time point of view, to path integrals, and eventually to Feynman diagrams and renormalization.

We will sketch very briefly how Feynman handled the forced oscillator, using the symbol S for the generalized action. He wrote

$$S = S_0 + \int dt \left\{ \frac{m\dot{x}^2}{2} - \frac{m\omega^2 x^2}{2} + \gamma(t)x \right\},$$

where S_0 is the action of the other particles of the system of which the oscillator $[x(t)]$ is a part, and $\gamma(t)x$ is the interaction of the oscillator with the particles that form the rest of the system. If $\gamma(t)$ is a simple function of time (for example $\cos \omega_1 t$) then it represents a given force applied to the oscillator. However, more generally we are dealing with an oscillator interacting with another quantum system and $\gamma(t)$ is a functional of the coordinates of that system. Since the action $S - S_0$ depends quadratically and linearly on $x(t)$, the path integrals

¹⁵ Feynman mentions in this connection Fermi’s influential article “Quantum theory of Radiation,” *Rev. Mod. Phys.* **4** (1932) pp. 87–132. In this paper, the result is assumed to hold; it was proven earlier by Fermi in “Sopra l’elettrodinamica quantistica,” *Rendiconti della R. Accademia Nazionali dei Lincei* **9** (1929) pp. 881–887.

over the paths of the oscillator can be performed when calculating the transition amplitude of the system from the initial time 0 to the final time T . With $x(0) = x$ and $x(T) = x'$, Feynman calls the function so obtained $G_\gamma(x, x'; T)$, obtaining finally the formula for the transition amplitude

$$\langle \chi_T | 1 | \psi_0 \rangle_S = \int \chi_T(Q_m, x) e^{\frac{i}{\hbar} S_0[\dots Q_i \dots]} G_\gamma(x, x'; T) \psi_0(Q_0, x') \\ \times dx dx' \frac{\sqrt{g} dQ_m \cdots \sqrt{g} dQ_0}{N_m \cdots N_1},$$

where the Q 's are the coordinates of the system other than the oscillator.

By using the last expression in the problem of particles interacting through an intermediate oscillator having $x(0) = \alpha$ and $x(T) = \beta$, Feynman shows that the expected value of a functional of the coordinates of the particles alone (such as a transition amplitude) can be obtained with a certain action that does not involve the oscillator coordinates, but only the constants α and β .¹⁶ This eliminates the oscillator from the dynamics of the problem. Various other initial and/or final conditions on the oscillator are shown to lead to a similar result. A brief section labeled "Conclusion" completes the thesis.

Laurie M. Brown

April 2005

The editor (LMB) thanks to Professor David Kiang for his invaluable assistance in copy-editing the retyped manuscript and checking the equations.

¹⁶ In the abstract at the end of the thesis this conclusion concerning the interaction of two systems is summarized as follows: "It is shown that in quantum mechanics, just as in classical mechanics, under certain circumstances the oscillator can be completely eliminated, its place being taken by a direct, but, in general, not instantaneous, interaction between the two systems."

THE PRINCIPLE OF LEAST ACTION IN QUANTUM MECHANICS

RICHARD P. FEYNMAN

Abstract

A generalization of quantum mechanics is given in which the central mathematical concept is the analogue of the action in classical mechanics. It is therefore applicable to mechanical systems whose equations of motion cannot be put into Hamiltonian form. It is only required that some form of least action principle be available.

It is shown that if the action is the time integral of a function of velocity and position (that is, if a Lagrangian exists), the generalization reduces to the usual form of quantum mechanics. In the classical limit, the quantum equations go over into the corresponding classical ones, with the same action function.

As a special problem, because of its application to electrodynamics, and because the results serve as a confirmation of the proposed generalization, the interaction of two systems through the agency of an intermediate harmonic oscillator is discussed in detail. It is shown that in quantum mechanics, just as in classical mechanics, under certain circumstances the oscillator can be completely eliminated, its place being taken by a direct, but, in general, not instantaneous, interaction between the two systems.

The work is non-relativistic throughout.

I. Introduction

Planck's discovery in 1900 of the quantum properties of light led to an enormously deeper understanding of the attributes and behaviour of matter, through the advent of the methods of quantum mechanics. When, however, these same methods are turned to the problem of light and the electromagnetic field great difficulties arise which have not been surmounted satisfactorily, so that Planck's observations still

remain without a consistent fundamental interpretation.¹

As is well known, the quantum electrodynamics that have been developed suffer from the difficulty that, taken literally, they predict infinite values for many experimental quantities which are obviously quite finite, such as for example, the shift in energy of spectral lines due to interaction of the atom and the field. The classical field theory of Maxwell and Lorentz serves as the jumping-off point for this quantum electrodynamics. The latter theory, however, does not take over the ideas of classical theory concerning the internal structure of the electron, which ideas are so necessary to the classical theory to attain finite values for such quantities as the inertia of an electron. The researches of Dirac into the quantum properties of the electron have been so successful in interpreting such properties as its spin and magnetic moment, and the existence of the positron, that is hard to believe that it should be necessary in addition to attribute internal structure to it.

It has become, therefore, increasingly more evident that before a satisfactory quantum electrodynamics can be developed it will be necessary to develop a classical theory capable of describing charges without internal structure. Many of these have now been developed, but we will concern ourselves in this thesis with the theory of action at a distance worked out in 1941 by J. A. Wheeler and the author.²

The new viewpoint pictures electrodynamic interaction as direct interaction at a distance between particles. The field then becomes a mathematical construction to aid in the solution of problems involving these interactions. The following principles are essential to the altered viewpoint:

(1) The acceleration of a point charge is due to the sum of its interactions with other charged particles. A charge does not act on itself.

¹ It is important to develop a satisfactory quantum electrodynamics also for another reason. At the present time theoretical physics is confronted with a number of fundamental unsolved problems dealing with the nucleus, the interactions of protons and neutrons, etc. In an attempt to tackle these, meson field theories have been set up in analogy to the electromagnetic field theory. But the analogy is unfortunately all too perfect; the infinite answers are all too prevalent and confusing.

² Not published. See, however, *Phys. Rev.* **59**, 683 (1941).

(2) The force of interaction which one charge exerts on another is calculated by means of the Lorentz force formula, $F = e[E + \frac{v}{c} \times H]$, in which the fields are the fields generated by the first charge according to Maxwell's equations.

(3) The fundamental (microscopic) phenomena in nature are symmetrical with respect to interchange of past and future. This requires that the solution of Maxwell's equation to be used in computing the interactions is to be half the retarded plus half the advanced solution of Lienard and Wiechert.

These principles, at first sight at such variance with elementary notions of causality, do in fact lead to essential agreement with the results of the more usual form of electrodynamics, and at the same time permit a consistent description of point charges and lead to a unique law of radiative damping. That this is the case has been shown in the work already referred to (see note 2). It is shown that these principles are equivalent to the equations of motion resulting from a principle of least action. The action function (due to Tetrode,³ and, independently, to Fokker⁴) involves only the coordinates of the particles, no mention of fields being made. The field is therefore a derived concept, and cannot be pictured as analogous to the vibrations of some medium, with its own degrees of freedom (for example, the energy density is not necessarily positive.) Perhaps a word or two as to what aspects of this theory make it a reasonable basis for a quantum theory of light would not be amiss.

When one attempts to list those phenomena which seem to indicate that light is quantized, the first type of phenomenon which comes to mind are like the photoelectric effect or the Compton effect. One is however, struck by the fact that since these phenomena deal with the interaction of light and matter their explanation may lie in the quantum aspects of matter, rather than requiring photons of light. This supposition is aided by the fact that if one solves the

³ H. Tetrode, *Zeits. f. Physik* **10**, 317 (1922).

⁴ A. D. Fokker, *Zeits. f. Physik* **38**, 386 (1929); *Physica* **9**, 33 (1929); *Physica* **12**, 145 (1932).

problem of an atom being perturbed by a potential varying sinusoidally with the time, which would be the situation if matter were quantum mechanical and light classical, one finds indeed that it will in all probability eject an electron whose energy shows an increase of $h\nu$, where ν is the frequency of variation of the potential. In a similar way an electron perturbed by the potential of two beams of light of different frequencies and different directions will make transitions to a state in which its momentum and energy is changed by an amount just equal to that given by the formulas for the Compton effect, with one beam corresponding in direction and wavelength to the incoming photon and the other to the outgoing one. In fact, one may correctly calculate in this way the probabilities of absorption and induced emission of light by an atom.

When, however, we come to spontaneous emission and the mechanism of the production of light, we come much nearer to the real reason for the apparent necessity of photons. The fact that an atom emits spontaneously at all is impossible to explain by the simple picture given above. In empty space an atom emits light and yet there is no potential to perturb the systems and so force it to make a transition. The explanation of modern quantum mechanical electrodynamics is that the atom is perturbed by the zero-point fluctuations of the quantized radiation field.

It is here that the theory of action at a distance gives us a different viewpoint. It says that an atom alone in empty space would, in fact, *not* radiate. Radiation is a consequence of the interaction with other atoms (namely, those in the matter which absorbs the radiation). We are then led to the possibility that the spontaneous radiation of an atom in quantum mechanics also, may not be spontaneous at all, but induced by the interaction with other atoms, and that all of the apparent quantum properties of light and the existence of photons may be nothing more than the result of matter interacting with matter directly, and according to quantum mechanical laws.

An attempt to investigate this possibility and to find a quantum analogue of the theory of action at a distance, meets first the difficulty

that it may not be correct to represent the field as a set of harmonic oscillators, each with its own degree of freedom, since the field in actuality is entirely determined by the particles. On the other hand, an attempt to deal quantum mechanically directly with the particles, which would seem to be the most satisfactory way to proceed, is faced with the circumstance that the equations of motion of the particles are expressed classically as a consequence of a principle of least action, and cannot, it appears, be expressed in Hamiltonian form.

For this reason a method of formulating a quantum analog of systems for which no Hamiltonian, but rather a principle of least action, exists has been worked out. It is a description of this method which constitutes this thesis. Although the method was worked out with the express purpose of applying it to the theory of action at a distance, it is in fact independent of that theory, and is complete in itself. Nevertheless most of the illustrative examples will be taken from problems which arise in the action at a distance electrodynamics. In particular, the problem of the equivalence in quantum mechanics of direct interaction and interaction through the agency of an intermediate harmonic oscillator will be discussed in detail. The solution of this problem is essential if one is going to be able to compare a theory which considers field oscillators as real mechanical and quantized systems, with a theory which considers the field as just a mathematical construction of classical electrodynamics required to simplify the discussion of the interactions between particles. On the other hand, no excuse need be given for including this problem, as its solution gives a very direct confirmation, which would otherwise be lacking, of the general utility and correctness of the proposed method of formulating the quantum analogue of systems with a least action principle.

The results of the application of these methods to quantum electrodynamics is not included in this thesis, but will be reserved for a future time when they shall have been more completely worked out.

It has been the purpose of this introduction to indicate the motivation for the problems which are discussed herein. It is to be emphasized again that the work described here is complete in itself without regard to its application to electrodynamics, and it is this circumstance which makes it appear advisable to publish these results as an independent paper. One should therefore take the viewpoint that the present paper is concerned with the problem of finding a quantum mechanical description applicable to systems which in their classical analogue are expressible by a principle of least action, and not necessarily by Hamiltonian equations of motion.

The thesis is divided into two main parts. The first deals with the properties of classical systems satisfying a principle of least action, while the second part contains the method of quantum mechanical description applicable to these systems. In the first part are also included some mathematical remarks about functionals. All of the analysis will apply to non-relativistic systems. The generalization to the relativistic case is not at present known.

II. Least Action in Classical Mechanics

1. The Concept of a Functional

The mathematical concept of a functional will play a rather predominant role in what is to follow so that it seems advisable to begin at once by describing a few of the properties of functionals and the notation used in this paper in connection with them. No attempt is made at mathematical rigor.

To say F is a functional of the function $q(\sigma)$ means that F is a number whose value depends on the *form* of the function $q(\sigma)$ (where σ is just a parameter used to specify the form of $q(\sigma)$). Thus,

$$F = \int_{-\infty}^{\infty} q(\sigma)^2 e^{-\sigma^2} d\sigma \quad (1)$$

is a functional of $q(\sigma)$ since it associates with every choice of the function $q(\sigma)$ a number, namely the integral. Also, the area under

a curve is a functional of the function representing the curve, since to each such function a number, the area is associated. The expected value of the energy in quantum mechanics is a functional of the wave function. Again,

$$F = q(0) \quad (2)$$

is a functional, which is especially simple because its value depends only on the value of the function $q(\sigma)$ at the one point $\sigma = 0$.

We shall write, if F is a functional of $q(\sigma)$, $F[q(\sigma)]$. A functional may have its argument more than one function, or functions of more than one parameter, as

$$F[x(t, s), y(t, s)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t, s)y(t, s) \sin \omega(t - s) dt ds.$$

A functional $F[q(\sigma)]$ may be looked upon as a function of an infinite number of variables, the variables being the value of the function $q(\sigma)$ at each point σ . If the interval of the range of σ is divided up into a large number of points σ_i , and the value of the function at these points is $q(\sigma_i) = q_i$, say, then approximately our functional may be written as a function of the variables q_i . Thus, in the case of equation (1) we could write, approximately,

$$F(\cdots q_i \cdots) = \sum_{i=-\infty}^{\infty} q_i^2 e^{-\sigma_i^2} (\sigma_{i+1} - \sigma_i).$$

We may define a process analogous to differentiation for our functionals. Suppose the function $q(\sigma)$ is altered slightly to $q(\sigma) + \lambda(\sigma)$ by the addition of a small function $\lambda(\sigma)$. From our approximate viewpoint we can say that each of the variables is changed from q_i to $q_i + \lambda_i$. The function is thereby changed by an amount

$$\sum_i \frac{\partial F(\cdots q_i \cdots)}{\partial q_i} \lambda_i.$$

In the case of a continuous number of variables, the sum becomes an integral and we may write, to the first order in λ ,

$$F[q(\sigma) + \lambda(\sigma)] - F[q(\sigma)] = \int K(t) \lambda(t) dt, \quad (3)$$

where $K(t)$ depends on F , and is what we shall call the functional derivative of F with respect to q at t , and shall symbolize, with Eddington,⁵ by $\frac{\delta F[q(\sigma)]}{\delta q(t)}$. It is not simply $\frac{\partial F(\dots q_i \dots)}{\partial q_i}$ as this is in general infinitesimal, but is rather the sum of these $\frac{\partial F}{\partial q_i}$ over a short range of i , say from $i+k$ to $i-k$, divided by the interval of the parameter, $\sigma_{i+k} - \sigma_{i-k}$.

Thus we write,

$$F[q(\sigma) + \lambda(\sigma)] = F[q(\sigma)] + \int \frac{\delta F[q(\sigma)]}{\delta q(t)} \lambda(t) dt + \text{higher order terms in } \lambda. \quad (4)$$

For example, in equation (1) if we substitute $q + \lambda$ for q , we obtain

$$\begin{aligned} F[q + \lambda] &= \int [q(\sigma)^2 + 2q(\sigma)\lambda(\sigma) + \lambda(\sigma)^2] e^{-\sigma^2} d\sigma \\ &= \int q(\sigma)^2 e^{-\sigma^2} d\sigma + 2 \int q(\sigma)\lambda(\sigma) e^{-\sigma^2} d\sigma \\ &\quad + \text{higher terms in } \lambda. \end{aligned}$$

Therefore, in this case, we have $\frac{\delta F[q]}{\delta q(t)} = 2q(t)e^{-t^2}$. In a similar way, if $F[q(\sigma)] = q(0)$, then $\frac{\delta F}{\delta q(t)} = \delta(t)$, where $\delta(t)$ is Dirac's delta symbol, defined by $\int \delta(t)f(t)dt = f(0)$ for any continuous function f .

The function $q(\sigma)$ for which $\frac{\delta F}{\delta q(t)}$ is zero for all t is that function for which F is an extremum. For example, in classical mechanics the action,

$$\mathcal{A} = \int L(\dot{q}(\sigma), q(\sigma)) d\sigma \quad (5)$$

is a functional of $q(\sigma)$. Its functional derivative is,

$$\frac{\delta \mathcal{A}}{\delta q(t)} = -\frac{d}{dt} \left\{ \frac{\partial L(\dot{q}(t), q(t))}{\partial \dot{q}} \right\} + \frac{\partial L(\dot{q}(t), q(t))}{\partial q}. \quad (6)$$

If $\mathcal{A}$ is an extremum the right hand side is zero.

⁵ A. S. Eddington, "The Mathematical Theory of Relativity" (1923) p. 139.

Editor's note: We have changed Eddington's symbol for the functional derivative to that now commonly in use.

2. The Principle of Least Action

For most mechanical systems it is possible to find a functional, $\mathcal{A}$, called the action, which assigns a number to each possible mechanical path, $q_1(\sigma), q_2(\sigma) \dots q_N(\sigma)$, (we suppose N degrees of freedom, each with a coordinate $q_n(\sigma)$, a function of a parameter (time) σ) in such a manner that this number is an extremum for an actual path $\bar{q}(\sigma)$ which could arise in accordance with the laws of motion. Since this extremum often is a minimum this is called the principle of least action. It is often convenient to use the principle itself, rather than the Newtonian equations of motion as the fundamental mechanical law. The form of the functional $\mathcal{A}[q_1(\sigma) \dots q_N(\sigma)]$ depends on the mechanical problem in question.

According to the principle of least action, then, if $\mathcal{A}[q_1(\sigma) \dots q_N(\sigma)]$ is the action functional, the equations of motion are N in number and are given by,

$$\frac{\delta \mathcal{A}}{\delta q_1(t)} = 0, \frac{\delta \mathcal{A}}{\delta q_2(t)} = 0, \dots, \frac{\delta \mathcal{A}}{\delta q_N(t)} = 0 \quad (7)$$

(We shall often simply write $\frac{\delta \mathcal{A}}{\delta q(t)} = 0$, as if there were only one variable). That is to say if all the derivatives of $\mathcal{A}$, with respect to $q_n(t)$, computed for the functions $\bar{q}_m(\sigma)$ are zero for all t and all n , then $\bar{q}_m(\sigma)$ describes a possible mechanical motion for the systems.

We have given an example, in equation (5), for the usual one dimensional problem when the action is the time integral of a Lagrangian (a function of position and velocity, only). As another example consider an action function arising in connection with the theory of action at a distance:

$$\mathcal{A} = \int_{-\infty}^{\infty} \left\{ \frac{m(\dot{x}(t))^2}{2} - V(x(t)) + k^2 \dot{x}(t) \dot{x}(t + T_0) \right\} dt. \quad (8)$$

It is approximately the action for a particle in a potential $V(x)$, and interacting with itself in a distant mirror by means of retarded and advanced waves. The time it takes for light to reach the mirror from the particle is assumed constant, and equal to $T_0/2$. The quantity

k^2 depends on the charge on the particle and its distance from the mirror. If we vary $x(t)$ by a small amount, $\lambda(t)$, the consequent variation in $\mathcal{A}$ is,

$$\begin{aligned}\delta\mathcal{A} &= \int_{-\infty}^{\infty} \{m\dot{x}(t)\dot{\lambda}(t) - V'(x(t))\lambda(t) + k^2\dot{\lambda}(t)\dot{x}(t + T_0) \\ &\quad + k^2\dot{\lambda}(t + T_0)\dot{x}(t)\}dt \\ &= \int_{-\infty}^{\infty} \{-m\ddot{x}(t) - V'(x(t)) - k^2\ddot{x}(t + T_0) \\ &\quad - k^2\ddot{x}(t - T_0)\}\lambda(t)dt, \quad \begin{array}{l} \text{by integrating} \\ \text{by part} \end{array}\end{aligned}$$

so that, according to our definition (4), we may write,

$$\frac{\delta\mathcal{A}}{\delta x(t)} = -m\ddot{x}(t) - V'(x(t)) - k^2\ddot{x}(t + T_0) - k^2\ddot{x}(t - T_0). \quad (9)$$

The equation of motion of this system is obtained, according to (7) by setting $\frac{\delta\mathcal{A}}{\delta x(t)}$ equal to zero. It will be seen that the force acting at time t depends on the motion of the particle at other time than t . The equations of motion cannot be described directly in Hamiltonian form.

3. Conservation of Energy. Constants of the Motion⁶

The problem we shall study in this section is that of determining to what extent the concepts of conservation of energy, momentum, etc., may be carried over to mechanical problems with a general form of action function. The usual principle of conservation of energy asserts that there is a function of positions at the time t , say, and of velocities of the particles whose value, for the actual motion of the particles, does not change with time. In our more general case however, the forces do not involve the positions of the particles only at one particular time, but usually a calculation of the forces requires

⁶ This section is not essential to an understanding of the remainder of the paper.

a knowledge of the paths of the particles over some considerable range of time (see for example, Eq. (9)). It is not possible in this case generally to find a constant of the motion which only involves the positions and velocities at one time.

For example, in the theory of action at a distance, the kinetic energy of the particles is not conserved. To find a conserved quantity one must add a term corresponding to the “energy in the field”. The field, however, is a functional of the motion of the particles, so that it is possible to express this “field energy” in terms of the motion of the particles. For our simple example (8), account of the equations of motion (9), the quantity,

$$E(t) = \frac{m(\dot{x}(t))^2}{2} + V(x(t)) - k^2 \int_t^{t+T_0} \ddot{x}(\sigma - T_0) \dot{x}(\sigma) d\sigma + k^2 \dot{x}(t) \dot{x}(t + T_0), \quad (10)$$

has, indeed, a zero derivative with respect to time. The first two terms represent the ordinary energy of the particles. The additional terms, representing the energy of interaction with the mirror (or rather, with itself) require a knowledge of the motion of the particle from the time $t - T_0$ to $t + T_0$.

Can we really talk about conservation, when the quantity conserved depends on the path of the particles over considerable ranges of time? If the force acting on a particle be $F(t)$ say, so that the particle satisfies the equation of motion $m\ddot{x}(t) = F(t)$, then it is perfectly clear that the integral,

$$I(t) = \int_{-\infty}^t [m\ddot{x}(t) - F(t)] \dot{x}(t) dt \quad (11)$$

has zero derivative with respect to t , when the path of the particle satisfies the equation of motion. Many such quantities having the same properties could easily be devised. We should not be inclined to say (11) actually represents a quantity of interest, in spite of its constancy.

The conservation of a physical quantity is of considerable interest because in solving problems it permits us to forget a great number of details. The conservation of energy can be derived from the laws of motion, but its value lies in the fact that by the use of it certain broad aspects of a problem may be discussed, without going into the great detail that is often required by a direct use of the laws of motion.

To compute the quantity $I(t)$, of equation (11), for two different times, t_1 and t_2 that are far apart, in order to compare $I(t_1)$ with $I(t_2)$, it is necessary to have detailed information of the path during the entire interval t_1 to t_2 . The value of I is equally sensitive to the character of the path for all times between t_1 and t_2 , even if these times lie very far apart. It is for this reason that the quantity $I(t)$ is of little interest. If, however, F were to depend on $x(t)$ only, so that it might be derived from a potential, (e.g.; $F = -V'(x)$), then the integrand is a perfect differential, and may be integrated to become $\frac{1}{2}m(\dot{x}(t))^2 + V(x(t))$. A comparison of I for two times, t_1 and t_2 , now depends only on the motion in the neighborhood of these times, all of the intermediate details being, so to speak, integrated out.

We therefore require two things if a quantity $I(t)$ is to attract our attention as being dynamically important. The first is that it be conserved, $I(t_1) = I(t_2)$. The second is that $I(t)$ should depend only locally on the path. That is to say, if one changes the path at some time t' in a certain (arbitrary) way, the change which is made in $I(t)$ should decrease to zero as t' gets further and further from t . That is to say, we should like the condition $\frac{\delta I(t)}{\delta q_n(t')} \rightarrow 0$ as $|t - t'| \rightarrow \infty$ satisfied.⁷

⁷ A more complete mathematical analysis than we include here is required to state rigorously just how fast it must approach zero as $|t - t'|$ approaches infinity. The proofs states herein are certainly valid if the quantities in (12) and (20) are assumed to become and remain equal to zero for values of $|t - t'|$ greater than some finite one, no matter how large it may be.

The energy expression (10) satisfies this criterion, as we have already pointed out. Under what circumstances can we derive an analogous constant of the motion for a general action function?

We shall, in the first place, impose a condition on the equations of motion which seems to be necessary in order that an integral of the motion of the required type exist. In the equation $\frac{\delta \mathcal{A}}{\delta q(t)} = 0$, which holds for an arbitrary time, t , we shall suppose that the influence of changing the path at time t' becomes less and less as $|t - t'|$ approaches infinity. That is to say, we require,

$$\frac{\delta^2 \mathcal{A}}{\delta q(t) \delta q(t')} \rightarrow 0 \quad \text{as} \quad |t - t'| \rightarrow \infty. \quad (12)$$

We next suppose that there exists a transformation (or rather, a continuous group of transformation) of coordinates, which we symbolize by $q_n \rightarrow q_n + x_n(a)$ and which leaves the action invariant (for example, the transformation may be a rotation). The transformation is to contain a parameter, a , and is to be a continuous function of a . For a equal to zero, the transformation should reduce to the identity, so that $x_n(0) = 0$. For very small a we may expand; $x_n(a) = 0 + ay_n + \dots$. That is to say, for infinitesimal a , if the coordinates q_n are changed to $q_n + ay_n$ the action is left unchanged;

$$\mathcal{A}[q_n(\sigma)] = \mathcal{A}[q_n(\sigma) + ay_n(\sigma)]. \quad (13)$$

For example, if the form of the action is unchanged if the particles take the same path at a later time, we may take, $q_n(t) \rightarrow q_n(t+a)$. In this case, for small a , $q_n(t) \rightarrow q_n(t) + a\dot{q}_n(t) + \dots$ so that $y_n = \dot{q}_n(t)$.

For each such continuous set of transformations there will be a constant of the motion. If the action is invariant with respect to change from $q(t)$ to $q(t+a)$, then an energy will exist. If the action is invariant with respect to the translation of all the coordinates (rectangular coordinates, that is) by the same distance, a , then a momentum in the direction of the translation may be derived. For rotations around an axis through the angle, the corresponding constant of the motion is the angular momentum around that axis. We

may show this connection between the groups of transformations and the constants of the motion, in the following way: For small a , from (13), we shall have,

$$\mathcal{A}[q_n(\sigma)] = \mathcal{A}[q_n(\sigma) + ay_n(\sigma)].$$

Expanding the left side with respect to the change in the coordinate $ay_n(\sigma)$, according to (4) to the first order in a we have,

$$\mathcal{A}[q_n(\sigma) + ay_n(\sigma)] = \mathcal{A}[q_n(\sigma)] + a \sum_{n=1}^N \int_{-\infty}^{\infty} y_n(t) \frac{\delta \mathcal{A}}{\delta q_n(t)} dt \quad (14)$$

and therefore, on account of (13),

$$\sum_{n=1}^N \int_{-\infty}^{\infty} y_n(\sigma) \frac{\delta \mathcal{A}}{\delta q_n(\sigma)} d\sigma = 0. \quad (15)$$

Now consider the quantity,

$$I(T) = \sum_{n=1}^N \int_{-\infty}^T y_n(\sigma) \frac{\delta \mathcal{A}}{\delta q_n(\sigma)} d\sigma. \quad (16)$$

On account of (15) we can also write,

$$I(T) = - \sum_{n=1}^N \int_T^{\infty} y_n(\sigma) \frac{\delta \mathcal{A}}{\delta q_n(\sigma)} d\sigma. \quad (17)$$

Consider the derivative of I with respect to T ; $\frac{dI(T)}{dT} = \sum_n y_n(T) \frac{\delta \mathcal{A}}{\delta q_n(T)}$ According to the equations of motion (7), this is seen to vanish, so that $I(T)$ is independent of T for the real motion, and is therefore conserved. We must now prove, in order that it be acceptable as an important constant of the motion, that

$$\frac{\delta I(T)}{\delta q_m(t)} \rightarrow 0 \text{ as } |T - t| \rightarrow \infty \text{ for any } m. \quad (18)$$

Suppose first that $t > T$. Let us compute $\frac{\delta I(T)}{\delta q_m(t)}$ directly from equation (16), obtaining,

$$\begin{aligned} \frac{\delta I(T)}{\delta q_m(t)} = & + \int_{-\infty}^T \sum_m \frac{\delta y_n(\sigma)}{\delta q_m(t)} \frac{\delta \mathcal{A}}{\delta q_m(\sigma)} d\sigma \\ & + \int_{-\infty}^T \sum_m y_n(\sigma) \frac{\delta^2 \mathcal{A}}{\delta q_m(t) \delta q_m(\sigma)} d\sigma. \end{aligned} \quad (19)$$

Now we shall suppose that $y_n(\sigma)$ does not depend very much on values of $q_m(t)$ for times, t , far away from σ . That is to say we shall assume,⁸

$$\frac{\delta y_n(\sigma)}{\delta q_m(t)} \rightarrow 0 \quad \text{as} \quad |\sigma - t| \rightarrow \infty. \quad (20)$$

In the first integral then, since $t > T$, and since only values of σ less than T appear in the integrand, for all such values, $t - \sigma > t - T$. As $t - T$ approaches infinity, therefore, only terms in the first integral of (19) for which $t - \sigma$ approaches infinity appear. We shall suppose that $\frac{\delta y_n(\sigma)}{\delta q_m(t)}$ decreases sufficiently rapidly with increase in $t - \sigma$ that the integral of it goes to zero as $t - T$ becomes infinite. A similar analysis applies to the second integral of (19). Here the quantity $\frac{\delta^2 \mathcal{A}}{\delta q_m(t) \delta q_m(\sigma)}$ approaches zero because of our assumption (12), and we shall suppose this approach sufficiently rapid that the integral vanish in the limit.

Thus we have shown that $\frac{\delta I(T)}{\delta q_m(t)} \rightarrow 0$ as $t - T \rightarrow \infty$. To prove the corresponding relation for $T - t \rightarrow \infty$ one may calculate $\frac{\delta I(T)}{\delta q_m(t)}$ with $t < T$ from (17), and proceed in exactly the same manner. In this way we can establish the required relation (18). This then shows that $I(T)$ is an important quantity which is conserved.

A particularly important example is, of course, the energy expression. This is got by the transformation of displacing the time, as has

⁸ In fact, for all practical cases which come to mind (energy momentum, angular momentum, corresponding to time displacement, translation, and rotation), $\frac{\delta y_n(\sigma)}{\delta q_m(t)}$ is actually zero if $\sigma \neq t$.

already been mentioned, for which $y_n(\sigma) = \dot{q}_n(\sigma)$. The energy integral may therefore be expressed, according to (16) (we have changed the sign), as,

$$E(T) = - \int_{-\infty}^T \sum_{n=1}^N \dot{q}_n(\sigma) \frac{\delta \mathcal{A}}{\delta q_n(\sigma)} d\sigma. \quad (21)$$

In our example (8) we would get from this formula,

$$E(T) = - \int_{-\infty}^T \dot{x}(\sigma) [-m\ddot{x}(\sigma) - V'(x(\sigma)) - k^2\ddot{x}(\sigma + T_0) - k^2\ddot{x}(\sigma - T_0)] d\sigma, \quad (22)$$

from which (10) has been derived by direct integration.

4. Particles Interacting through an Intermediate Oscillator

The problem we are going to discuss in this section, since it will give us a good example of a system for which only a principle of least action exists, is the following: Let us suppose we have two particles A and B which do not interact directly with each other, but there is a harmonic oscillator, O with which both of the particles A and B interact. The harmonic oscillator, therefore serves as an intermediary by means of which particle A is influenced by the motion of particle B and vice versa. In what way is this interaction through the intermediate oscillator equivalent to a direct interaction between the particles A and B , and can the motion of these particles, A , B , be expressed by means of a principle of least action, not involving the oscillator? (In the theory of electrodynamics this is the problem as to whether the interaction of particles through the intermediary of the field oscillators can also be expressed as a direct interaction at a distance.)

To make the problem precise, we let $y(t)$ and $z(t)$ represent coordinates of the particles A and B at the time t . Let the Lagrangians of the particles alone be designated by L_y and L_z . Let them each interact with the oscillator (with coordinate $x(t)$) Lagrangian

$\frac{m}{2}(\dot{x}^2 - \omega^2 x^2)$) by means of a term in the Lagrangian for the entire system, which is of the form $(I_y + I_z)x$, where I_y is a function involving the coordinates of atom A only, and I_z is some function of the coordinates of B . (We have assumed the interaction linear in the coordinate of the oscillator.)

We then ask: If the action integral for y, z, x , is

$$\int \left[L_y + L_z + \left(\frac{m\dot{x}^2}{2} - \frac{m\omega^2 x^2}{2} \right) + (I_y + I_z)x \right] dt, \quad (23)$$

is it possible to find an action $\mathcal{A}$, a functional of $y(t), z(t)$, only, such that, as far as the motion of the particles A, B , are concerned, (i.e., for variations of $y(t), z(t)$) the action $\mathcal{A}$ is a minimum?

In the first place, since the actual motion of the particles A, B , depends not only on y, z , initially (or at any other time) but also on the initial conditions satisfied by the oscillator, it is clear that $\mathcal{A}$ is not determined absolutely, but the form that $\mathcal{A}$ takes must have some dependence on the state of the oscillator.

In the second place, since we are interested in an action principle for the particles, we must consider variations of the motion of these particles from the true motion. That is, we must consider dynamically impossible paths for these particles. We thus meet a new problem; when varying the motion of the particle A and B , what do we do about the oscillator? We cannot keep the entire motion of the oscillator fixed, for that would require having this entire motion directly expressed in the action integral and we should be back where we started, with the action (23).

The answer to this question lies in the observation made above that the action must involve somehow some of the properties of the oscillator. In fact, since the oscillator has one degree of freedom it will require two numbers (e.g. position and velocity) to specify the state of the oscillator sufficiently accurately that the motion of the particles A and B is uniquely determined. Therefore in the action function for these particles, two parameters enter, which are arbitrary, and represent some properties of the motion of the oscillator. When the

variation in the motion of the particles is taken, these quantities must be considered as constants, and thus, it is the properties of the oscillator described by these quantities which are considered fixed for the “impossible” motion of the particles. We should not be surprised to find that the action function for the particles depends on which properties of the oscillator are considered to be held fixed in the variation of y , and z . It is, on the other hand, somewhat unexpected that, as we shall see, not all possible conditions on the oscillator give rise to motions of y and z which are simply expressible in terms of an action principle. Let us see how this works out in detail.

We shall assume that the functions I_y and I_z are zero for times t greater than T and less than 0. We shall also assume (for simplicity only) that I_y is a function of t and $y(t)$ only, and does not depend on $\dot{y}(t)$. Similarly, I_z is to be independent of $\dot{z}(t)$. Then, from (23), the equation of motion of particle y is,

$$\frac{d}{dt} \left(\frac{\partial L_y}{\partial \dot{y}} \right) - \frac{\partial L_y}{\partial y} = \frac{\partial I_y}{\partial y} x(t), \quad (24)$$

with a similar equation for z . That for the oscillator is,

$$m\ddot{x} + m\omega^2 x = [I_y(t) + I_z(t)]. \quad (25)$$

The solution of this last equation is, where we have written $\gamma = I_y + I_z$

$$x(t) = x(0) \cos \omega t + \dot{x}(0) \frac{\sin \omega t}{\omega} + \frac{1}{m\omega} \int_0^t \gamma(s) \sin \omega(t-s) ds. \quad (26)$$

This may be expressed in other ways, for example,

$$\begin{aligned} x(t) = & \frac{\sin \omega(T-t)}{\sin \omega T} \left[x(0) - \frac{1}{m\omega} \int_0^t \sin \omega s \gamma(s) ds \right] \\ & + \frac{\sin \omega t}{\sin \omega T} \left[x(T) - \frac{1}{m\omega} \int_t^T \sin \omega(T-s) \gamma(s) ds \right], \end{aligned} \quad (27)$$

or again,

$$\begin{aligned}
 x(t) = & \frac{1}{\sin \omega T} [R_T \sin \omega t + R_0 \sin \omega (T - t)] \\
 & + \frac{1}{2m\omega} \int_0^t \sin \omega (t - s) \gamma(s) ds \\
 & - \frac{1}{2m\omega} \int_t^T \sin \omega (t - s) \gamma(s) ds, \quad (28)
 \end{aligned}$$

where we have written

$$R_0 = \frac{1}{2} \left[x(0) + x(T) \cos \omega T - \dot{x}(T) \frac{\sin \omega T}{\omega} \right] \quad (29)$$

and

$$R_T = \frac{1}{2} \left[x(T) + x(0) \cos \omega T - \dot{x}(0) \frac{\sin \omega T}{\omega} \right]. \quad (30)$$

It is seen that R_T is the mean of the coordinate of the oscillator at time T and what that coordinate would have been at this time if the oscillator had been free and started with its actual initial conditions. Similarly, R_0 is the mean of the initial coordinate and what that coordinate would have had to be, were the oscillator free, to produce the actual final conditions at time T . Outside the time range 0 to T the oscillator is, of course, simply a free oscillator.

These expressions for $x(t)$ may be substituted into (24) and the corresponding equation for z , to obtain various expressions for the motion of the particles y, z ; each expressed in terms of these particles and two parameters connected with the oscillator. For the expression (26) these parameters are $x(0)$ and $\dot{x}(0)$; for (27) they are $x(0)$ and $x(T)$; for (28) they are R_0 and R_T . We should like to determine whether these expressions could be obtained from an action for y and z only.

If the action be $\mathcal{A}$, then the expression (24) must be of the form $\frac{\delta \mathcal{A}}{\delta y(t)} = 0$. That is to say, we must have,

$$\frac{\delta \mathcal{A}}{\delta y(t)} = -\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}} \right) + \frac{\partial L}{\partial y} \Big|_t + \frac{\partial I_y}{\partial y} \Big|_t \cdot x(t). \quad (31)$$

We seek the solution of this expression for each expression we may write for $x(t)$.

Now an equation such as (31) (which is really an infinite set of equations, one for each value of t) does not always have a solution. One of the necessary requirements is, since $\frac{\delta}{\delta y(s)} \left(\frac{\delta \mathcal{A}}{\delta y(t)} \right) = \frac{\delta}{\delta y(t)} \left(\frac{\delta \mathcal{A}}{\delta y(s)} \right)$, that,

$$\begin{aligned} \frac{\delta}{\delta y(s)} \left[-\frac{d}{dt} \left(\frac{\partial L_y}{\partial \dot{y}} \right) + \frac{\partial L_y}{\partial y} \Big|_t + \frac{\partial I_y}{\partial y} \Big|_t \cdot x(t) \right] \\ = \frac{\delta}{\delta y(t)} \left[-\frac{d}{ds} \left(\frac{\partial L_y}{\partial \dot{y}} \right) + \frac{\partial L_y}{\partial y} \Big|_s + \frac{\partial I_y}{\partial y} \Big|_s \cdot x(s) \right]. \end{aligned}$$

This requires, therefore, if $s \neq t$, that $x(t)$ satisfy,

$$\frac{\partial I_y}{\partial y} \Big|_t \frac{\delta x(t)}{\delta y(s)} = \frac{\partial I_y}{\partial y} \Big|_s \frac{\delta x(s)}{\delta y(t)}. \quad (32)$$

For the expression (26), we have,

$$\begin{aligned} \frac{\delta x(t)}{\delta y(s)} &= \frac{1}{m\omega} \sin \omega(t-s) \cdot \frac{\partial I_y}{\partial y} \Big|_s \quad \text{if } s < t \\ &= 0 \quad \text{if } s > t. \end{aligned} \quad (33)$$

With this expression equation (32) is not satisfied so that we may conclude that no simple action function will describe the motion of the particles A and B if the initial position and velocity of the oscillator are considered as fixed.