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AN INTRODUCTION TO
BLACK HOLES, INFORMATION and the
STRING THEORY REVOLUTION

The Holographic Universe

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AN INTRODUCTION TO BLACK HOLES, INFORMATION, AND THE STRING THEORY REVOLUTION

The Holographic Universe

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Preface

It is now almost a century since the year 1905, in which the principle of relativity and the hypothesis of the quantum of radiation were introduced. It has taken most of that time to synthesize the two into the modern quantum theory of fields and the standard model of particle phenomena. Although there is undoubtedly more to be learned both theoretically and experimentally, it seems likely that we know most of the basic principles which follow from combining the special theory of relativity with quantum mechanics. It is unlikely that a major revolution will spring from this soil.

By contrast, in the 80 years that we have had the general theory of relativity, nothing comparable has been learned about the quantum theory of gravitation. The methods that were invented to quantize electrodynamics, which were so successfully generalized to build the standard model, prove wholly inadequate when applied to gravitation. The subject is riddled with paradox and contradiction. One has the distinct impression that we are thinking about the things in the wrong way. The paradigm of relativistic quantum field theory almost certainly has to be replaced.

How then are we to go about finding the right replacement? It seems very unlikely that the usual incremental increase of knowledge from a combination of theory and experiment will ever get us where we want to go, that is, to the Planck scale. Under this circumstance our best hope is an examination of fundamental principles, paradoxes and contradictions, and the study of gedanken experiments. Such strategy has worked in the past. The earliest origins of quantum mechanics were not experimental atomic physics, radioactivity, or spectral lines. The puzzle which started the whole thing was a contradiction between the principles of statistical thermodynamics and the field concept of Faraday and Maxwell. How was it possible, Planck asked, for the infinite collection of radiation oscillators to have a finite specific heat?

In the case of special relativity it was again a conceptual contradiction and a gedanken experiment which opened the way. According to Einstein, at the age of 15 he formulated the following paradox: suppose an observer moved along with a light beam and observed it. The electromagnetic field would be seen as a static, spatially varying field. But no such solution to Maxwell's equations exists. By this simple means a contradiction was exposed between the symmetries of Newton's and Galileo's mechanics and those of Maxwell's electrodynamics.

The development of the general theory from the principle of equivalence and the man-in-the-elevator gedanken experiment is also a matter of historical fact. In each of these cases the consistency of readily observed properties of nature which had been known for many years required revolutionary paradigm shifts.

What known properties of nature should we look to, and which paradox is best suited to our present purposes? Certainly the most important facts are the success of the general theory in describing gravity and of quantum mechanics in describing the microscopic world. Furthermore, the two theories appear to lead to a serious clash that once again involves statistical thermodynamics in an essential way. The paradox was discovered by Jacob Bekenstein and turned into a serious crisis by Stephen Hawking. By an analysis of gedanken experiments, Bekenstein realized that if the second law of thermodynamics was not to be violated in the presence of a black hole, the black hole must possess an intrinsic entropy. This in itself is a source of paradox. How and why a classical solution of field equations should be endowed with thermodynamical attributes has remained obscure since Bekenstein's discovery in 1972.

Hawking added to the puzzle when he discovered that a black hole will radiate away its energy in the form of Planckian black body radiation. Eventually the black hole must completely evaporate. Hawking then raised the question of what becomes of the quantum correlations between matter outside the black hole and matter that disappears behind the horizon. As long as the black hole is present, one can do the bookkeeping so that it is the black hole itself which is correlated to the matter outside. But eventually the black hole will evaporate. Hawking then made arguments that there is no way, consistent with causality, for the correlations to be carried by the outgoing evaporation products. Thus, according to Hawking, the existence of black holes inevitably causes a loss of quantum coherence and breakdown of one of the basic principles of quantum mechanics – the evolution of pure states to pure states. For two decades this contradiction between

the principles of general relativity and quantum mechanics has so puzzled theorists that many now see it as a serious crisis.

Hawking and much of the traditional relativity community have been of the opinion that the correct resolution of the paradox is simply that quantum coherence is lost during black hole evaporation. From an operational viewpoint this would mean that the standard rules of quantum mechanics would not apply to processes involving black holes. Hawking further argued that once the loss of quantum coherence is permitted in black hole evaporation, it becomes compulsory in all processes involving the Planck scale. The world would behave as if it were in a noisy environment which continuously leads to a loss of coherence. The trouble with this is that there is no known way to destroy coherence without, at the same time violating energy conservation by heating the world. The theory is out of control as argued by Banks, Peskin and Susskind, and 't Hooft. Throughout this period, a few theorists, including 't Hooft and Susskind, have felt that the basic principles of quantum mechanics and statistical mechanics have to be made to co-exist with black hole evaporation.

't Hooft has argued that by resolving the paradox and removing the contradiction, the way to the new paradigm will be opened. The main purpose of this book is to lay out this case.

A second purpose involves development of string theory as a unified description of elementary particles, including their gravitational interactions. Although still very incomplete, string theory appears to be a far more consistent mathematical framework for quantum gravity than ordinary field theory. It is therefore worth exploring the differences between string theory and field theory in the context of black hole paradoxes. Quite apart from the question of the ultimate correctness and consistency of string theory, there are important lessons to be drawn from the differences between these two theories. As we shall see, although string theory is usually well approximated by local quantum field theory, in the neighborhood of a black hole horizon the differences become extreme. The analysis of these differences suggests a resolution of the black hole dilemma and a completely new view of the relations between space, time, matter, and information.

The quantum theory of black holes, with or without strings, is far from being a textbook subject with well defined rules. To borrow words from Sidney Coleman, it is a “trackless swamp” with many false but seductive paths and no maps. To navigate it without disaster we will need some beacons in the form of trusted principles that we can turn to for direction. In this book the absolute truth of the following four propositions will be

assumed: 1) The formation and evaporation of a black hole is consistent with the basic principles of quantum mechanics. In particular, this means that observations performed by observers who remain outside the black hole can be described by a unitary time evolution. The global process, beginning with asymptotic infalling objects and ending with asymptotic outgoing evaporation products is consistent with the existence of a unitary S-matrix. 2) The usual semiclassical description of quantum fields in a slowly varying gravitational background is a good approximation to certain coarse grained features of the black hole evolution. Those features include the thermodynamic properties, luminosity, energy momentum flux, and approximate black body character of Hawking radiation. 3) Thirdly we assume the usual connection between thermodynamics and quantum statistical mechanics. Thermodynamics results from coarse graining a more microscopic description so that states with similar macroscopic behavior are lumped into a single thermodynamic state. The existence of a thermodynamics will be taken to mean that a microscopic set of degrees of freedom exists whose coarse graining leads to the thermal description. More specifically we assume that a thermodynamic entropy S implies that approximately $\exp(S)$ quantum states have been lumped into one thermal state.

These three propositions, taken by themselves, are in no way radical. Proposition 1 and 3 apply to all known forms of matter. Proposition 2 may perhaps be less obvious, but it nevertheless rests on well-established foundations. Once we admit that a black hole has energy, entropy, and temperature, it must also have a luminosity. Furthermore the existence of a thermal behavior in the vicinity of the horizon follows from the equivalence principle as shown in the fundamental paper of Unruh. Why then should any of these principles be considered controversial? The answer lies in a fourth proposition which seems as inevitable as the first three: 4) The fourth principle involves observers who fall through the horizon of a large massive black hole, carrying their laboratories with them. If the horizon scale is large enough so that tidal forces can be ignored, then a freely falling observer should detect nothing out of the ordinary when passing the horizon. The usual laws of nature with no abrupt external perturbations will be found valid until the influence of the singularity is encountered. In considering the validity of this fourth proposition it is important to keep in mind that the horizon is a global concept. The existence, location, size, and shape of a horizon depend not only on past occurrences, but also on future events. We ourselves could right now be at the horizon of a gigantic black

hole caused by matter yet to collapse in the future. The horizon in classical relativity is simply the mathematical surface which separates those points from which any light ray must hit a singularity from those where light may escape to infinity. A mathematical surface of this sort should have no local effect on matter in its vicinity.

In Chapter 9 we will encounter powerful arguments against the mutual consistency of propositions 1–4. The true path through the swamp at times becomes so narrow it seems to be a dead end, while all around false paths beckon. Beware the will-o'-the-wisp and don't lose your nerve.

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PART 1

Black Holes and Quantum Mechanics

Chapter 1

The Schwarzschild Black Hole

Before beginning the study of the quantum theory of black holes, one must first become thoroughly familiar with the geometry of classical black holes in a variety of different coordinate systems. Each coordinate system that we will study has its own particular utility, and no one of them is in any sense the best or most correct description. For example, the Kruskal–Szekeres coordinate system is valuable for obtaining a global overview of the entire geometry. It can however be misleading when applied to observations made by distant observers who remain outside the horizon during the entire history of the black hole. For these purposes, Schwarzschild coordinates, or the related tortoise coordinates, which cover only the exterior of the horizon are in many ways more valuable.

We begin with the simplest spherically symmetric static uncharged black holes described by Schwarzschild geometry.

1.1 Schwarzschild Coordinates

In Schwarzschild coordinates, the Schwarzschild geometry is manifestly spherically symmetric and static. The metric is given by

$$\begin{aligned} d\tau^2 &= \left(1 - \frac{2MG}{r}\right) dt^2 - \left(1 - \frac{2MG}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 \\ &= g_{\mu\nu} dx^\mu dx^\nu. \end{aligned} \tag{1.1.1}$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$.

The coordinate t is called Schwarzschild time, and it represents the time recorded by a standard clock at rest at spatial infinity. The coordinate r is called the Schwarzschild radial coordinate. It does not measure proper

spatial distance from the origin, but is defined so that the area of the 2-sphere at r is $4\pi r^2$. The angles θ, ϕ are the usual polar and azimuthal angles. In equation 1.1.1 we have chosen units such that the speed of light is 1.

The horizon, which we will tentatively define as the place where g_{00} vanishes, is given by the coordinate $r = 2MG$. At the horizon g_{rr} becomes singular. The question of whether the geometry is truly singular at the horizon or if it is the choice of coordinates which are pathological is subtle. In what follows we will see that no local invariant properties of the geometry are singular at $r = 2MG$. Thus a small laboratory in free fall at $r = 2MG$ would record nothing unusual. Nevertheless there is a very important sense in which the horizon is **globally** special if not singular. To a distant observer the horizon represents the boundary of the world, or at least that part which can influence his detectors.

To determine whether the local geometry is singular at $r = 2MG$ we can send an explorer in from far away to chart it. For simplicity let's consider a radially freely falling observer who is dropped from rest from the point $r = R$. The trajectory of the observer in parametric form is given by

$$r = \frac{R}{2}(1 + \cos\eta) \quad (1.1.2)$$

$$\tau = \frac{R}{2} \left(\frac{R}{2MG} \right)^{1/2} (\eta + \sin\eta) \quad (1.1.3)$$

$$t = \left(\frac{R}{2} + 2MG \right) \left(\frac{R}{2MG} - 1 \right)^{1/2} \eta + \frac{R}{2} \left(\frac{R}{2MG} - 1 \right)^{1/2} \sin\eta \\ + 2MG \log \left| \frac{\left(\frac{R}{2MG} - 1 \right)^{1/2} + \tan \frac{\eta}{2}}{\left(\frac{R}{2MG} - 1 \right)^{1/2} - \tan \frac{\eta}{2}} \right| \quad [0 < \eta < \pi] \quad (1.1.4)$$

where τ is the proper time recorded by the observer's clock. From these overly complicated equations it is not too difficult to see that the observer arrives at the point $r = 0$ after a finite interval

$$\tau = \frac{\pi}{2} R \left(\frac{R}{2MG} \right)^{\frac{1}{2}} \quad (1.1.5)$$

Evidently the proper time when crossing the horizon is finite and smaller than the expression in equation 1.1.5.

What does the observer encounter at the horizon? An observer in free fall is not sensitive to the components of the metric, but rather senses the tidal forces or curvature components. Define an orthonormal frame such that the observer is momentarily at rest. We can construct unit basis vectors, $\hat{\tau}$, $\hat{\rho}$, $\hat{\theta}$, $\hat{\phi}$ with $\hat{\tau}$ oriented along the observer's instantaneous time axis, and $\hat{\rho}$ pointing radially out. The non-vanishing curvature components are given by

$$\begin{aligned} R_{\hat{\tau}\hat{\theta}\hat{\tau}\hat{\theta}} &= R_{\hat{\tau}\hat{\phi}\hat{\tau}\hat{\phi}} = -R_{\hat{\rho}\hat{\theta}\hat{\rho}\hat{\theta}} = -R_{\hat{\rho}\hat{\phi}\hat{\rho}\hat{\phi}} = \frac{MG}{r^3} \\ R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} &= -R_{\hat{\tau}\hat{\rho}\hat{\tau}\hat{\rho}} = \frac{2MG}{r^3} \end{aligned} \quad (1.1.6)$$

Thus all the curvature components are finite and of order

$$R(\text{Horizon}) \sim \frac{1}{M^2 G^2} \quad (1.1.7)$$

at the horizon. For a large mass black hole they are typically very small. Thus the infalling observer passes smoothly and safely through the horizon.

On the other hand the tidal forces diverge as $r \rightarrow 0$ where a true local singularity occurs. At this point the curvature increases to the point where the classical laws of nature must fail.

Let us now consider the history of the infalling observer from the viewpoint of a distant observer. We may suppose that the infalling observer sends out signals which are received by the distant observer. The first surprising thing we learn from equations 1.1.2, 1.1.3, and 1.1.4 is that the crossing of the horizon does not occur at any finite Schwarzschild time. It is easily seen that as r tends to $2MG$, t tends to infinity. Furthermore a signal originating at the horizon cannot reach any point $r > 2MG$ until an infinite Schwarzschild time has elapsed. This is shown in Figure 1.1. Assuming that the infalling observer sends signals at a given frequency ν , the distant observer sees those signals with a progressively decreasing frequency. Over the entire span of Schwarzschild time the distant observer records only a finite number of pulses from the infalling transmitter. Unless the infalling observer increases the frequency of his/her signals to infinity as the horizon is approached, the distant observer will inevitably run out of signals and lose track of the transmitter after a finite number of pulses. The limits imposed on the information that can be transmitted from near the horizon are not so severe in classical physics as they are in quantum theory. According to classical physics the infalling observer can use an arbitrarily large carrier frequency to send an arbitrarily large amount of information using

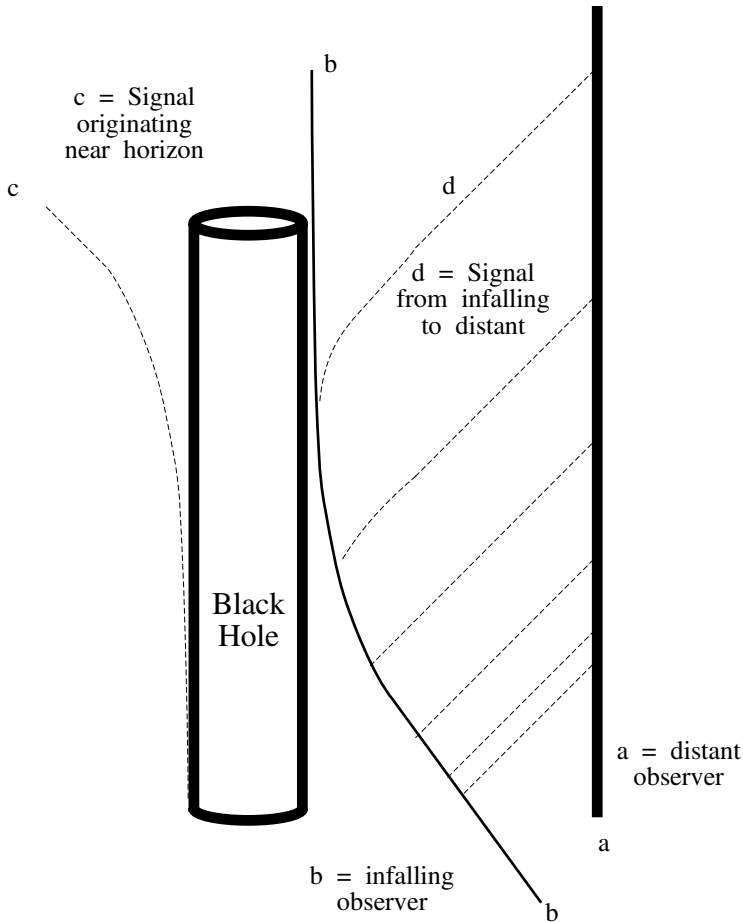


Fig. 1.1 *Infalling observer sending signals to distant Schwarzschild observer*

an arbitrarily small energy without significantly disturbing the black hole and its geometry. Therefore, in principle, the distant observer can obtain information about the neighborhood of the horizon and the infalling system right up to the point of horizon crossing. However quantum mechanics requires that to send even a single bit of information requires a quantum of energy. As the observer approaches the horizon, this quantum must have higher and higher frequency, implying that the observer must have had a large energy available. This energy will back react on the geometry, dis-

turbing the very quantity to be measured. Thereafter, as we shall see, no information can be transmitted from behind the horizon.

1.2 Tortoise Coordinates

A change of radial coordinate maps the horizon to minus infinity so that the resulting coordinate system covers only the region $r > 2MG$. We define the tortoise coordinate r^* by

$$\frac{1}{1 - \frac{2MG}{r}} dr^2 = \left(1 - \frac{2MG}{r}\right) (dr^*)^2 \quad (1.2.8)$$

so that

$$d\tau^2 = \left(1 - \frac{2MG}{r}\right) [dt^2 - (dr^*)^2] - r^2 d\Omega^2 \quad (1.2.9)$$

The interesting point is that the radial-time part of the metric now has a particularly simple form, called *conformally flat*. A space is called conformally flat if its metric can be brought to the form

$$d\tau^2 = F(x) dx^\mu dx^\nu \eta_{\mu\nu} \quad (1.2.10)$$

with $\eta_{\mu\nu}$ being the usual Minkowski metric. Any two-dimensional space is conformally flat, and a slice through Schwarzschild space at fixed θ, ϕ is no exception. In equation 1.2.9 the metric of such a slice is manifestly conformally flat. Furthermore it is also static.

The tortoise coordinate r^* is given explicitly by

$$r^* = r + 2MG \log \left(\frac{r - 2MG}{2MG} \right) \quad (1.2.11)$$

Note: $r^* \rightarrow -\infty$ at the horizon.

We shall see that wave equations in the black hole background have a very simple form in tortoise coordinates.

.....

1.3 Near Horizon Coordinates (Rindler space)

The region near the horizon can be explored by replacing r by a coordinate ρ which measures proper distance from the horizon:

$$\begin{aligned}\rho &= \int_{2MG}^r \sqrt{g_{rr}(r')} dr' \\ &= \int_{2MG}^r \left(1 - \frac{2MG}{r'}\right)^{-\frac{1}{2}} dr' \\ &= \sqrt{r(r-2MG)} + 2MG \sinh^{-1}\left(\sqrt{\frac{r}{2MG}-1}\right)\end{aligned}\quad (1.3.12)$$

In terms of ρ and t the metric takes the form

$$d\tau^2 = \left(1 - \frac{2MG}{r(\rho)}\right) dt^2 - d\rho^2 - r(\rho)^2 d\Omega^2 \quad (1.3.13)$$

Near the horizon equation 1.3.12 behaves like

$$\rho \approx 2\sqrt{2MG(r-2MG)} \quad (1.3.14)$$

giving

$$d\tau^2 \cong \rho^2 \left(\frac{dt}{4MG}\right)^2 - d\rho^2 - r^2(\rho) d\Omega^2 \quad (1.3.15)$$

Furthermore, if we are interested in a small angular region of the horizon arbitrarily centered at $\theta = 0$ we can replace the angular coordinates by Cartesian coordinates

$$x = 2MG\theta \cos\phi \quad (1.3.16)$$

$$y = 2MG\theta \sin\phi$$

Finally, we can introduce a dimensionless time ω

$$\omega = \frac{t}{4MG} \quad (1.3.17)$$

and the metric then takes the form

$$d\tau^2 = \rho^2 d\omega^2 - d\rho^2 - dx^2 - dy^2 \quad (1.3.18)$$

It is now evident that ρ and ω are radial and hyperbolic angle variables for an ordinary Minkowski space. Minkowski coordinates T, Z can be

defined by

$$T = \rho \sinh \omega \quad (1.3.19)$$

$$Z = \rho \cosh \omega$$

to get the familiar Minkowski metric

$$d\tau^2 = dT^2 - dZ^2 - dX^2 - dY^2 \quad (1.3.20)$$

It should be kept in mind that equation 1.3.20 is only accurate near $r = 2MG$, and only for a small angular region. However it clearly demonstrates that the horizon is locally nonsingular, and, for a large black hole, is locally almost indistinguishable from flat space-time.

In Figure 1.2 the relation between Minkowski coordinates and the ρ, ω coordinates is shown. The entire Minkowski space is divided into four quadrants labeled I, II, III, and IV. Only one of those regions, namely

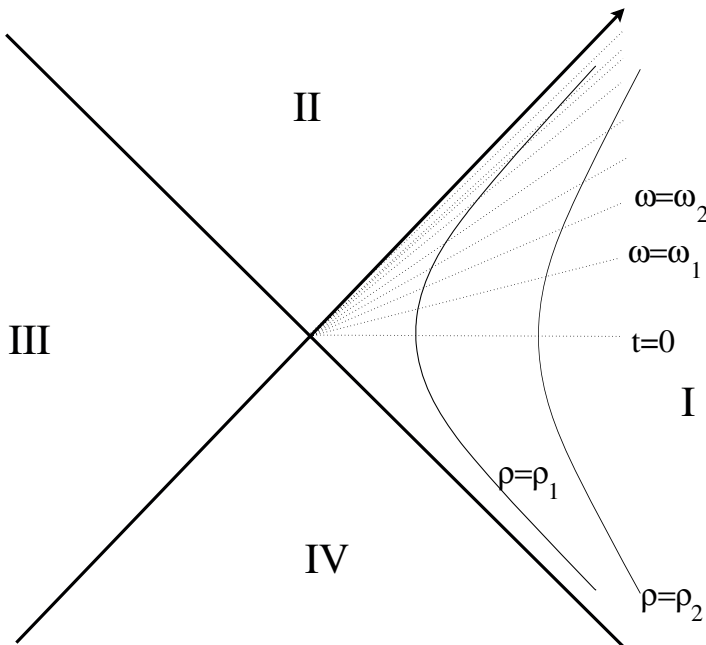


Fig. 1.2 Relation between Minkowski and Rindler coordinates

Region I lies outside the black hole horizon. The horizon itself is the origin $T = Z = 0$. Note that it is a two-dimensional surface in the four-dimensional space-time. This may appear surprising, since originally the horizon was defined by the single constraint $r = 2MG$, and therefore appears to be a three dimensional surface. However, recall that at the horizon g_{00} vanishes. Therefore the horizon has no extension or metrical size in the time direction.

The approximation of the near-horizon region by Minkowski space is called the Rindler approximation. In particular the portion of Minkowski space approximating the exterior region of the black hole, i.e. Region I, is called Rindler space. The time-like coordinate, ω , is called Rindler time. Note that a translation of Rindler time $\omega \rightarrow \omega + \text{constant}$ is equivalent to a Lorentz boost in Minkowski space.

1.4 Kruskal–Szekeres Coordinates

Finally we can bring the black hole metric to the form

$$d\tau^2 = F(R) [R^2 d\omega^2 - dR^2] - r^2 d\Omega^2 \quad (1.4.21)$$

For small ρ equation 1.3.15 shows that $\rho \approx R$. A more accurate comparison with the original Schwarzschild metric gives the following requirements:

$$R^2 F(R) = 16M^2 G^2 \left[1 - \frac{2MG}{r} \right] \quad (1.4.22)$$

$$F(R) dR^2 = \frac{1}{1 - \frac{2MG}{r}} dr^2 \quad (1.4.23)$$

from which it follows that

$$4MG \log \frac{R}{MG} = r + 2MG \log \left(\frac{r - 2MG}{2MG} \right) = r^* \quad (1.4.24)$$

or

$$R = MG \exp \left(\frac{r^*}{4MG} \right) \quad (1.4.25)$$

R and ω can be thought of as radial and hyperbolic angular coordinates of a space which is conformal to flat 1+1 dimensional Minkowski space.

Letting

$$\begin{aligned} R e^{\omega} &= V \\ R e^{-\omega} &= -U \end{aligned} \quad (1.4.26)$$

be “radial light-like” variables, the radial-time part of the metric takes the form

$$d\tau^2 = F(R) dU dV \quad (1.4.27)$$

The coordinates U, V are shown in Figure 1.3. The surfaces of constant r are the timelike hyperbolas in Figure 1.3. As r tends to $2MG$ the hyperbolas become the broken straight lines H^+ and H^- which we will call the extended past and future horizons. Although the extended horizons lie at finite values of the Kruskal–Szekeres coordinates, they are located at Schwarzschild time $\pm\infty$. Thus we see that a particle trajectory which crosses H^+ in a finite proper time, crosses $r = 2MG$ only after an infinite Schwarzschild time.

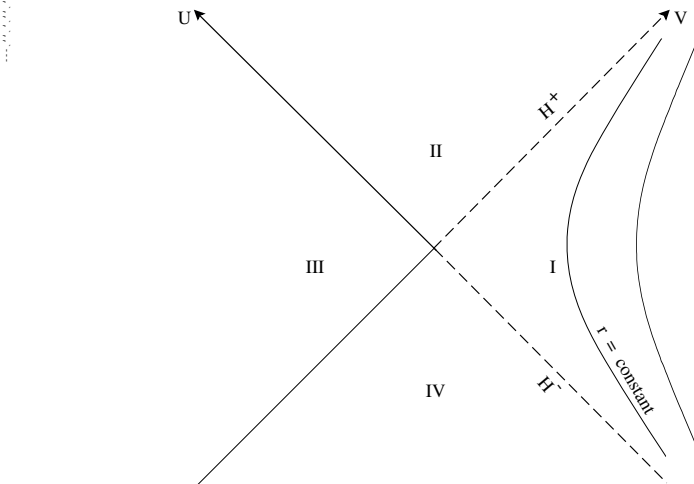


Fig. 1.3 U, V Kruskal–Szekeres coordinates

The region of Schwarzschild space with $r < 2MG$ can be taken to be Region II. In this region the surfaces of constant r are the

spacelike hyperboloids

$$UV = \text{positive constant} \quad (1.4.28)$$

The true singularity at $r = 0$ occurs at $R^2 = -(MG)^2$, or

$$UV = (MG)^2 \quad (1.4.29)$$

The entire maximal analytic extension of the Schwarzschild geometry is easily described in Kruskal–Szekeres coordinates. It is shown in Figure 1.4.

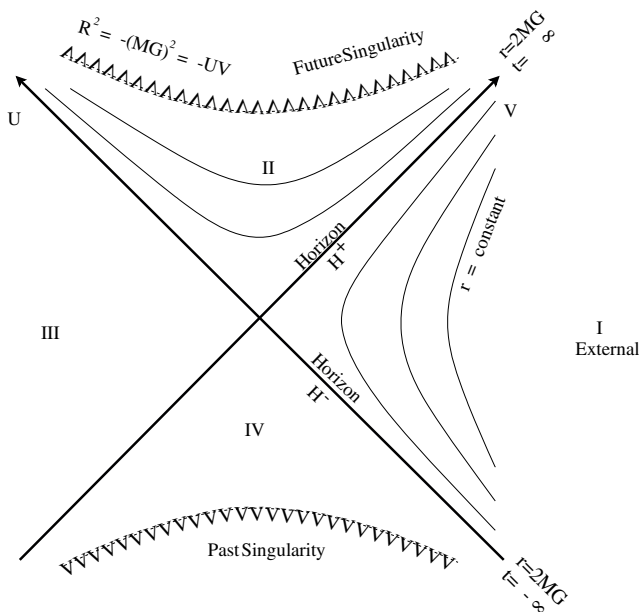


Fig. 1.4 *Maximal analytic extension of Schwarzschild in Kruskal–Szekeres coordinates*

A useful property of Kruskal–Szekeres coordinates is the fact that light rays and timelike trajectories always lie within a two-dimensional light cone bounded by 45° lines. A radial moving light ray travels on a trajectory $V = \text{constant}$ or $U = \text{constant}$. A nonradially directed light ray or timelike trajectory always lies inside the two-dimensional light cone. With this in mind, it is easy to understand the causal properties of the black hole

geometry. Consider a point P_1 in Region I. A radially outgoing light ray from P_1 will escape falling into the singularity as shown in Figure 1.5. An incoming light ray from P_1 will eventually cross H^+ and then hit the future singularity. Thus an observer in Region I can send messages to infinity as well as into Region II.

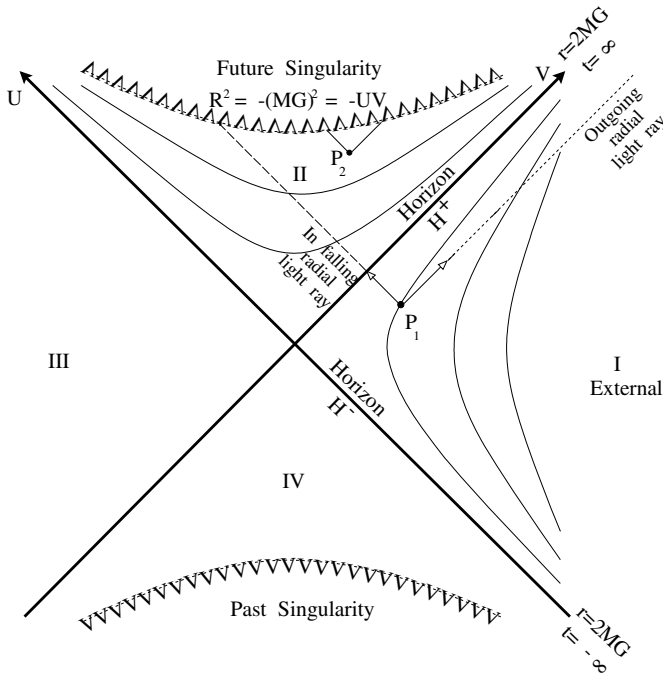


Fig. 1.5 Radial light rays from a point in Region I and Region II using Kruskal-Szekeres coordinates

Consider next Region II. From any point P_2 any signal must eventually hit the singularity. Furthermore, no signal can ever escape to Region I. Thus no observer who stays outside $r = 2MG$ can ever be influenced by events in Region II. For this reason Region II is said to be behind the horizon. Regions III and IV, as we will see, are not relevant to the classical problem of black holes formed by collapsing matter. Nevertheless let us consider them. From Region III no signal can ever get to Region I, and so it is also behind the horizon. On the other hand, points in Region IV can

communicate with Region I. Region I however cannot communicate with Region IV. All of this is usually described by saying that Regions II and III are behind the future horizon while Regions III and IV are behind the past horizon.

1.5 Penrose Diagrams

Penrose diagrams are a useful way to represent the causal structure of spacetimes, especially if, like the Schwarzschild black hole, they have spherical symmetry. They represent the geometry of a two-dimensional surface of fixed angular coordinates. Furthermore they “compactify” the geometry so that it can be drawn in total on the finite plane. As an example, consider ordinary flat Minkowski space. Ignoring angular coordinates,

$$d\tau^2 = dt^2 - dr^2 - \text{angular part} = (dt + dr)(dt - dr) - \text{angular part} \quad (1.5.30)$$

Radial light rays propagate on the light cone $dt \pm dr = 0$.

Any transformation that is of the form

$$\begin{aligned} Y^+ &= F(t + r) \\ Y^- &= F(t - r) \end{aligned} \quad (1.5.31)$$

will preserve the form of the light cone. We can use such a transformation to map the entire infinite space $0 \leq r \leq \infty$, $-\infty \leq t \leq +\infty$ to a finite portion of the plane. For example

$$\begin{aligned} Y^+ &= \tanh(t + r) \\ Y^- &= \tanh(t - r) \end{aligned} \quad (1.5.32)$$

The entire space-time is mapped to the finite triangle bounded by

$$\begin{aligned} Y^+ &= 1 \\ Y^- &= -1 \\ Y^+ - Y^- &= 0 \end{aligned} \quad (1.5.33)$$

as shown in Figure 1.6. Also shown in Figure 1.6 are some representative contours of constant r and t .

There are several infinities on the Penrose diagram. Future and past time-like infinities ($t = \pm\infty$) are the beginnings and ends of time-like trajectories. Space-like infinity ($r = \infty$) is where all space-like surfaces end. In addition to these there are two other infinities which are called $\mathcal{I}^\pm$.

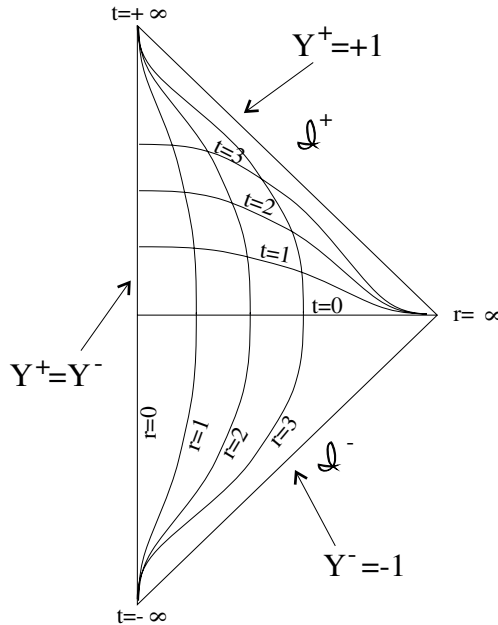


Fig. 1.6 Penrose diagram for Minkowski space

They are past and future light-like infinity, and they represent the origin of incoming light rays and the end of outgoing light rays.

Similar deformations can be carried out for more interesting geometries, such as the black hole geometry represented by Kruskal–Szekeres coordinates. The resulting Penrose diagram is shown in Figure 1.7.

1.6 Formation of a Black Hole

The eternal black hole described by the static Schwarzschild geometry is an idealization. In nature, black holes are formed from the collapse of gravitating matter. The simplest model for black hole formation involves a collapsing thin spherical shell of massless matter. For example, a shell of photons, gravitons, or massless neutrinos with very small radial extension and total energy M provides an example.

To construct the geometry, we begin with the empty space Penrose diagram with the infalling shell represented by an incoming light-like line

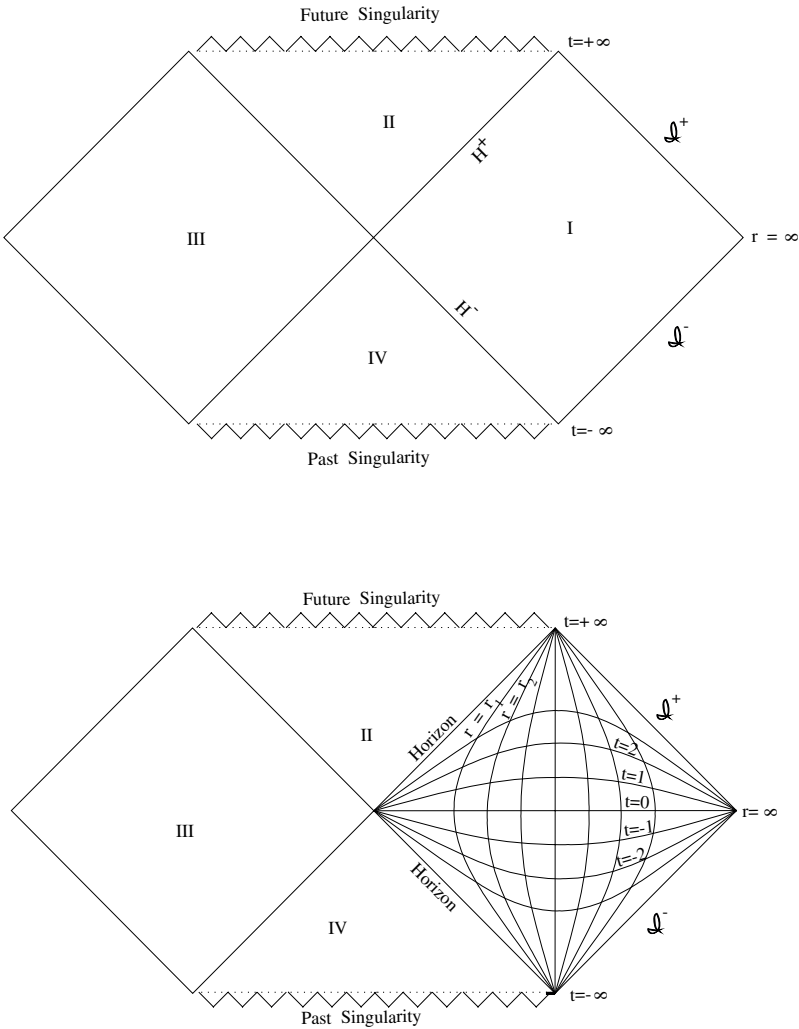


Fig. 1.7 Penrose diagram for Schwarzschild black hole, showing regions (top) and curves of fixed radial position and constant time (bottom)

(see Figure 1.8). The particular value of Y^+ chosen for the trajectory is arbitrary since any two such values are related by a time translation. The infalling shell divides the Penrose diagram into two regions, A and B. The Region A is interior to the shell and represents the initial flat space-time

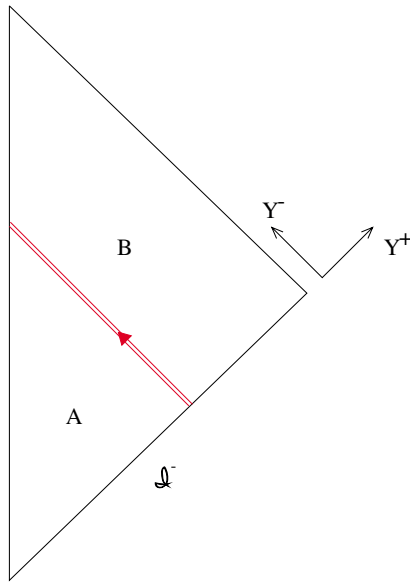


Fig. 1.8 *Minkowski space Penrose diagram for radially infalling spherical shell of massless particles with energy M*

before the shell passes. Region B is the region outside the shell and must be modified in order to account for the gravitational field due to the mass M .

In Newtonian physics the gravitational field exterior to a spherical mass distribution is uniquely that of a point mass located at the center of the distribution. Much the same is true in general relativity. In this case Birkoff's theorem tells us that the geometry outside the shell must be the Schwarzschild geometry. Accordingly, we consider the Penrose diagram for a black hole of mass M divided into regions A' and B' by an infalling massless shell as in Figure 1.9. Once again the particular value of Y^+ chosen for the trajectory is immaterial. Just as in Figure 1.8 where the Region B is unphysical, in Figure 1.9 the Region A' is to be discarded. To form the full classical evolution the regions A of Figure 1.8 and B' of Figure 1.9 must be glued together. However this must be done so that the "radius" of the local two sphere represented by the angular coordinates (θ, ϕ) is continuous. In other words, the mathematical identification of the boundaries of A and B' must respect the continuity of the variable r .

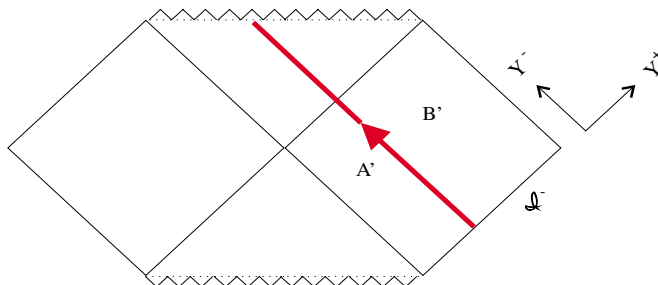


Fig. 1.9 Penrose diagram of Schwarzschild black hole with radially infalling shell of massless particle with energy M

Since in both cases r varies monotonically from $r = \infty$ at $\mathcal{I}^-$ to $r = 0$, the identification is always possible. One of the two Penrose diagrams will have to undergo a deformation along the Y^- direction in order to make the identification smoothly, but this will not disturb the form of the light cones. Thus in Figure 1.10 we show the resulting Penrose diagram for the complete geometry. On Fig 1.10, a light-like surface H is shown as a dotted line. It is clear that any light ray or timelike trajectory that originates to the upper left of H must end at the singularity and cannot escape to $\mathcal{I}^+$ (or $t = \infty$). This identifies H as the horizon. In Region B' the horizon is identical to the surface H^+ of Figure 1.7, that is it coincides with the future horizon of the final black hole geometry and is therefore found at $r = 2MG$. On the other hand, the horizon also extends into the Region A where the metric is just that of flat space-time. In this region the value of r on the horizon grows from an initial value $r = 0$ to the value $r = 2MG$ at the shell.

It is evident from this discussion that the horizon is a global and not a local concept. In the Region A no local quantity will distinguish the presence of the horizon whose occurrence is due entirely to the future collapse of the shell.

Consider next a distant observer located on a trajectory with $r \gg 2MG$. The observer originates at past time-like infinity and eventually ends at future time-like infinity, as shown in Figure 1.11. The distant observer collects information that arrives at any instant from his backward light cone. Evidently such an observer never actually sees events on the

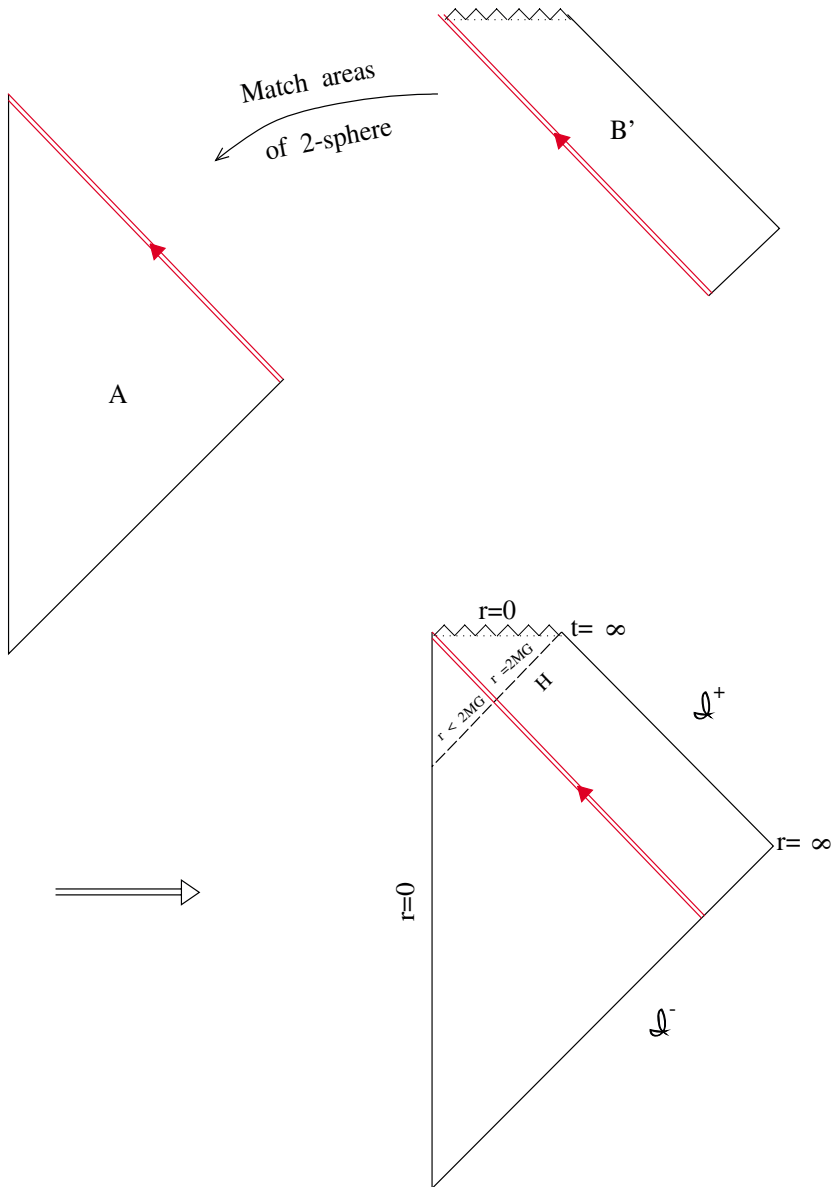


Fig. 1.10 Penrose diagram for collapsing shell of massless particles

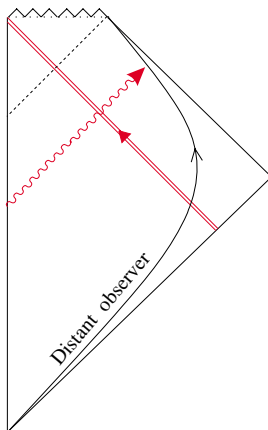


Fig. 1.11 *Distant observer to collapsing spherical shell*

horizon. In this sense the horizon must be regarded as at the end of time. Any particle or wave which falls through the horizon is seen by the distant observer as asymptotically approaching the horizon as it is infinitely red shifted. At least that is the case classically.

This basic description of black hole formation is much more general than might be guessed. It applies with very little modification to the collapse of all kinds of massive matter as well as to non-spherical distributions. In all cases the horizon is a lightlike surface which separates the space-time into an inner and an outer region. Any light ray which originates in the inner region can never reach future asymptotic infinity, or for that matter ever reach any point of the outer region. The events in the outer region can send light rays to $\mathcal{I}^+$ and time-like trajectories to $t = \infty$.

The horizon, as we have seen, is a global concept whose location depends on all future events. It is composed of a family of light rays or null geodesics, passing through each space-time point on the horizon. This is shown in Figure 1.12. Notice that null geodesics are vertical after the shell crosses the horizon and essentially at 45° prior to that crossing. These light rays are called the *generators* of the horizon.

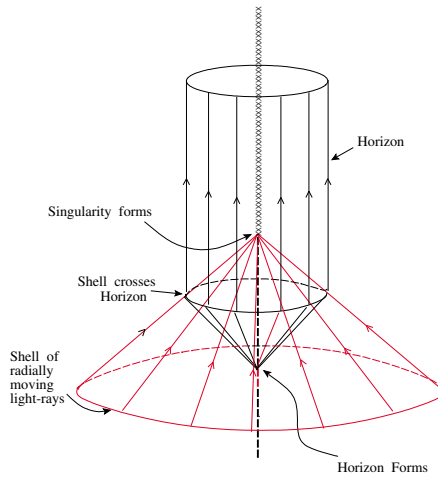


Fig. 1.12 The horizon as family of null geodesics

1.7 Fidos and Frefos and the Equivalence Principle

In considering the description of events near the horizon of a **static** black hole from the viewpoint of an external observer[1], it is useful to imagine space to be filled with static observers, each located at a fixed (r, θ, ϕ) . Such observers are called fiducial observers, or by the whimsical abbreviation, FIDOS. Each Fido carries a clock which may be adjusted to record Schwarzschild time t . This means that Fidos at different r values see their own clocks running at different proper rates. Alternatively, they could carry standard clocks which always record proper time τ . At a given r the relation between Schwarzschild time t and the Fidos proper time τ is given by

$$\frac{d\tau}{dt} = \sqrt{g_{00}} = \left[1 - \frac{2MG}{r}\right]^{\frac{1}{2}} \quad (1.7.34)$$

Thus, to the Fido near $r = 2MG$, the Schwarzschild clock appears to run at a very rapid rate. Another possible choice of clocks would record the dimensionless hyperbolic angle ω defined by equation 1.3.17.

The spatial location of the Fidos can be labeled by the angular coordinates (θ, ϕ) and any one of the radial variables r , r^* , or ρ . Classically the Fidos can be thought of as mathematical fictions or real but arbitrarily light

systems suspended by arbitrarily light threads from some sort of suspension system built around the black hole at a great distance. The acceleration of a Fido at proper distance ρ is given by $\frac{1}{\rho}$ for $\rho \ll MG$. Quantum mechanically we have a dilemma if we try to imagine the Fidos as real. If they are extremely light their locations will necessarily suffer large quantum fluctuations, and they will not be useful as fixed anchors labeling space-time points. If they are massive they will influence the gravitational field that we wish to describe. Quantum mechanically, physical Fidos must be replaced by a more abstract concept called *gauge fixing*. The concept of gauge fixing in gravitation theory implies a mathematical restriction on the choice of coordinates. However all real observables are required to be gauge invariant.

Now let us consider a classical particle falling radially into a black hole. There are two viewpoints we can adopt toward the description of the particle's motion. The first is the viewpoint of the Fidos who are permanently stationed outside the black hole. It is a viewpoint which is also useful to a distant observer, since any observation performed by a Fido can be communicated to distant observers. According to this viewpoint, the particle never crosses the horizon but asymptotically approaches it. The second viewpoint involves freely falling observers (FREFOS) who follow the particle as it falls. According to the Frefos, they and the particle cross the horizon after a finite time. However, once the horizon is crossed, their observations cannot be communicated to any Fido or to a distant observer.

Once the infalling particle is near the horizon its motion can be described by the coordinates (T, Z, X, Y) defined in equations 1.3.16 and 1.3.19. Since the particle is freely falling, in the Minkowski coordinates its motion is a straight line

$$\begin{aligned}\frac{dZ}{d\tau} &= \frac{p_Z}{m} = -\frac{p_Z}{m} \\ \frac{dT}{d\tau} &= \frac{p_T}{m}\end{aligned}\tag{1.7.35}$$

where p_Z and p_T are the Z and T components of momentum, and m is the mass of the particle. As the particle freely falls past the horizon, the components p_Z and p_T may be regarded as constant or slowly varying. They are the components seen by Frefos.

The components of momentum seen by Fidos are the components p_ρ and p_τ which, using equation 1.3.19, are given by

$$\begin{aligned}p_\rho &= p_Z \cosh\omega + p_T \sinh\omega \\ p_\tau &= p_Z \sinh\omega + p_T \cosh\omega\end{aligned}\tag{1.7.36}$$

For large times we find

$$p_\rho \approx p_\tau \approx 2p_Z \exp \omega = 2p_Z \exp\left(\frac{t}{4MG}\right) \quad (1.7.37)$$

Thus we find the momentum of an infalling particle as seen by a Fido grows exponentially with time! It is also easily seen that ρ , the proper spatial distance of the particle from the horizon, exponentially decreases with time

$$\rho(t) \approx \rho(0) \exp\left(-\frac{t}{4MG}\right) \quad (1.7.38)$$

Locally the relation between the coordinates of the Frefos and Fidos is a time dependent boost along the radial direction. The hyperbolic boost angle is the dimensionless time ω . Eventually, during the lifetime of the black hole this boost becomes so large that the momentum of an infalling particle (as seen by a Fido) quickly exceeds the entire mass of the universe.

As a consequence of the boost, the Fidos see all matter undergoing Lorentz contraction into a system of arbitrarily thin “pancakes” as it approaches the horizon. According to classical physics, the infalling matter is stored in “sedimentary” layers of diminishing thickness as it eternally sinks toward the horizon (see Figure 1.13). Quantum mechanically we must expect this picture to break down by the time the infalling particle has been squeezed to within a Planck distance from the horizon. The Frefos of course see the matter behaving in a totally unexceptional way.

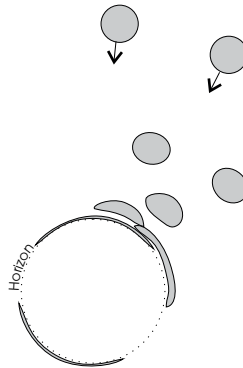


Fig. 1.13 *Sedimentary layers of infalling matter on horizon*

Chapter 2

Scalar Wave Equation in a Schwarzschild Background

In Chapters 3 and 4 we will be concerned with the behavior of quantum fields near horizons. In this lecture we will study the properties of a scalar wave equation in the background of a black hole.

Let us consider a conventional massless free Klein–Gordon field χ in the Schwarzschild background. Here we will find great advantage in utilizing tortoise coordinates in which the metric has the form

$$d\tau^2 = F(r^*) [dt^2 - (dr^*)^2] - r^2 [d\theta^2 + \sin^2\theta d\phi^2] \quad (2.0.1)$$

The action for χ is

$$\begin{aligned} I &= \frac{1}{2} \int \sqrt{-g} g^{\mu\nu} \partial_\mu \chi \partial_\nu \chi d^4x \\ &= \frac{1}{2} \int dt dr^* d\theta d\phi \left\{ \frac{(\partial_t \chi)^2 - (\partial_{r^*} \chi)^2}{F} \right. \\ &\quad \left. - \frac{1}{r^2} \left(\frac{\partial \chi}{\partial \theta} \right)^2 - \frac{1}{r^2 \sin^2 \theta} \left(\frac{\partial \chi}{\partial \phi} \right)^2 \right\} F r^2 \sin \theta \end{aligned} \quad (2.0.2)$$

Now define

$$\psi = r \chi \quad (2.0.3)$$

and the action takes the form

$$\begin{aligned} I &= \frac{1}{2} \int [(\partial_t \psi)^2 - \left(\frac{\partial \psi}{\partial r^*} - \frac{\partial(\ln r)}{\partial r^*} \psi \right)^2 \\ &\quad - \frac{F}{r^2} (\sin \theta \left(\frac{\partial \psi}{\partial \theta} \right)^2 + \frac{1}{\sin \theta} \left(\frac{\partial \psi}{\partial \phi} \right)^2)] dt dr^* d\theta d\phi \end{aligned} \quad (2.0.4)$$

which, after an integration by parts and the introduction of spherical harmonic decomposition becomes

$$I = \sum_{\ell m} \frac{1}{2} \int [(\dot{\psi}_{\ell m})^2 - \left(\frac{\partial \psi_{\ell m}}{\partial r^*} \right)^2 + \left\{ \left(\frac{\partial \ln r}{\partial r^*} \right)^2 + \frac{\partial}{\partial r^*} \left(\frac{\partial \ln r}{\partial r^*} \right) \right\} \psi_{\ell m}^2 - \frac{F}{r^2} \ell(\ell+1) \psi_{\ell m}^2] dt dr^* \quad (2.0.5)$$

Using the relation between r and r^*

$$r^* = r + 2MG \ln(r - 2MG)$$

gives for each ℓ, m an action

$$I_{\ell m} = \frac{1}{2} \int dt dr^* \left[\left(\frac{\partial \psi_{\ell m}}{\partial t} \right)^2 - \left(\frac{\partial \psi_{\ell m}}{\partial r^*} \right)^2 - V_{\ell}(r^*) \psi_{\ell m}^2 \right] \quad (2.0.6)$$

where the potential $V_{\ell}(r^*)$ is given by

$$V_{\ell}(r^*) = \frac{r - 2MG}{r} \left(\frac{\ell(\ell+1)}{r^2} + \frac{2MG}{r^3} \right) \quad (2.0.7)$$

The equation of motion is

$$\ddot{\psi}_{\ell m} = \frac{\partial^2 \psi_{\ell m}}{(\partial r^*)^2} - V_{\ell}(r^*) \psi_{\ell m} \quad (2.0.8)$$

and for a mode of frequency ν

$$-\frac{\partial^2 \psi_{\ell m}}{(\partial r^*)^2} + V_{\ell}(r^*) \psi_{\ell m} = \nu^2 \psi_{\ell m} \quad (2.0.9)$$

The potential V is shown in Figure 2.1 as a function of the Schwarzschild coordinate r . For $r \gg 3MG$ the potential is repulsive. In fact it is just the relativistic generalization of the usual repulsive centrifugal barrier. However as the horizon is approached, gravitational attraction wins and the potential becomes attractive, and pulls a wave packet toward the horizon. The maximum of the potential, where the direction of the force changes, depends weakly on the angular momentum ℓ . It is given by

$$r_{max} = 3MG \left(\frac{1}{2} \left(1 + \sqrt{1 + \frac{14\ell^2 + 14\ell + 9}{9\ell^2(\ell+1)^2}} \right) - \frac{1}{2\ell(\ell+1)} \right) \quad (2.0.10)$$

For $\ell \rightarrow \infty$ the maximum occurs at $r_{max}(\ell \rightarrow \infty) = 3MG$.

The same potential governs the motion of massless classical particles. One can see that the points $r_{max}(\ell)$ represent unstable circular orbits, and

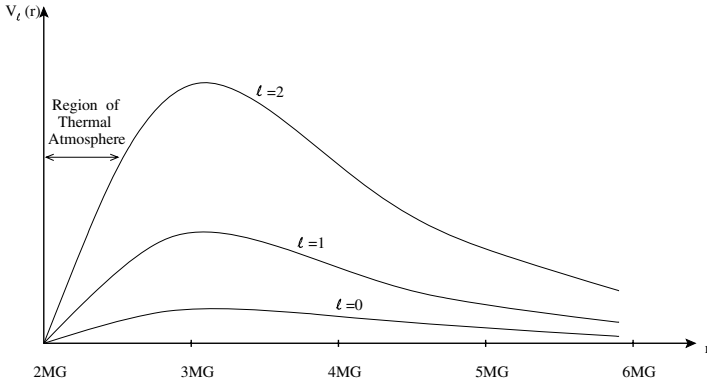


Fig. 2.1 *Effective potential for free scalar field vs Schwarzschild radial coordinate*

the innermost such orbit is at $r = 3MG$. Any particle that starts with vanishing radial velocity in the region $r < 3MG$ will spiral into the horizon.

In the region of large negative r^* where we approach the horizon, the potential is unimportant, and the field behaves like a free massless Klein–Gordon field. The eigenmodes in this region have the form of plane waves which propagate with unit velocity

$$\frac{dr^*}{dt} = \mp 1 \quad (2.0.11)$$

$$\psi \rightarrow e^{ik(r^* \pm t)}$$

Let us consider a field quantum of frequency ν and angular momentum ℓ propagating from large negative r^* toward the barrier at $r \approx 3MG$. Will it pass over the barrier? To answer this we note that equation 2.0.9 has the form of a Schrodinger equation for a particle of energy ν^2 in a potential V . The particle has enough energy to overcome the barrier without tunneling if ν^2 is larger than the maximum height of the barrier. For example, if $\ell = 0$ the height of the barrier is

$$V_{max} = \frac{1}{2M^2G^2} \left(\frac{3}{8}\right)^3 \quad (2.0.12)$$

An s-wave quantum will therefore escape if

$$\nu > \frac{0.15}{MG} \quad (2.0.13)$$

Similarly an s-wave quantum with $\nu > \frac{0.15}{MG}$ will be able to penetrate the barrier from the outside and fall to the horizon. Less energetic particles must tunnel through the barrier.

A particle of high angular momentum, whether on the inside or outside of the barrier will have more difficulty penetrating through. For large ℓ

$$V_{max} \approx \frac{1}{27} \frac{\ell^2}{M^2 G^2} \quad (2.0.14)$$

Therefore the threshold energy for passing over the barrier is

$$\nu \sim \frac{1}{\sqrt{27}} \frac{\ell}{MG} \quad (2.0.15)$$

2.1 Near the Horizon

Near the horizon the exterior of the black hole may be described by the Rindler metric

$$d\tau = \rho^2 d\omega^2 - d\rho^2 - dX^2 - dY^2$$

It is useful to replace ρ by a tortoise-like coordinate which again goes to $-\infty$ at the horizon. We define

$$u = \log \rho \quad (2.1.16)$$

and the metric near the horizon becomes

$$d\tau^2 = \exp(2u) [d\omega^2 - du^2] - dX^2 - dY^2 \quad (2.1.17)$$

The scalar field action becomes

$$I = \frac{1}{2} \int dX dY du d\omega \left[\left(\frac{\partial \chi}{\partial \omega} \right)^2 - \left(\frac{\partial \chi}{\partial u} \right)^2 - e^{2u} (\partial_{\perp} \chi)^2 \right] \quad (2.1.18)$$

where $\partial_{\perp} \chi = (\partial_X, \partial_Y)$. Instead of using spherical waves, near the horizon we can decompose χ into transverse plane waves with transverse wave vector $k_{\perp}$

$$\chi = \int d^2 k_{\perp} e^{i k_{\perp} \cdot x_{\perp}} \chi(k_{\perp}, u, \omega) \quad (2.1.19)$$

the action for a given wave number k is

$$I = \frac{1}{2} \int d\omega du \left[(\partial_\omega \chi)^2 - (\partial_u \chi)^2 - k^2 e^{2u} \chi^2 \right] \quad (2.1.20)$$

Thus the potential is

$$V(k, u) = k^2 e^{2u} \quad (2.1.21)$$

The correspondence between the momentum vector k and the angular momentum ℓ is given by the usual connection between momentum and angular momentum. If the horizon has circumference $2\pi(2MG)$, then a wave with wave vector $k_\perp$ will correspond to an angular momentum $|\ell| = |k| r = 2MG|k|$. Thus the potential in equation 2.1.21 is seen to be proportional to ℓ^2 . For very low angular momentum the approximation is not accurate, but qualitatively is correct for $\ell > 0$. In approximating a sum over ℓ and m by an integral over k , the integral should be infrared cut off at $|k| \sim \frac{1}{MG}$.

From the action in equation 2.1.20 we obtain the equation of motion

$$\frac{\partial^2 \chi}{\partial \omega^2} - \frac{\partial^2 \chi}{\partial u^2} + k^2 \exp(2u) \chi = 0 \quad (2.1.22)$$

A solution which behaves like $\exp(i\nu t)$ in Schwarzschild time has the form

$$e^{i\nu[4MG\omega]} = e^{i\lambda\omega} \quad (2.1.23)$$

The time independent form of the equation of motion is

$$-\frac{\partial^2 \chi}{\partial u^2} + (k^2 \exp(2u)) \chi = \lambda^2 \chi \quad (2.1.24)$$

Once again we see that unless $k = 0$, there is a potential confining quanta to the region near the horizon. Qualitatively, the behavior of a quantum field in a black hole background differs from the Rindler space approximation in that for the black hole, the potential barrier is cut off when $\rho = e^u$ is greater than MG . By contrast, in the Rindler case V increases as e^u without bound.

We have not thus far paid attention to the boundary conditions at the horizon where $u \rightarrow -\infty$. Since in this region the field $\chi(u)$ behaves like a free massless field, the boundary condition would be expected to be that the field is in the usual quantum ground state. In the next section we will see that this is not so.

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Chapter 3

Quantum Fields in Rindler Space

According to Einstein, the study of a phenomenon in a gravitational field is best preceded by a study of the same phenomenon in an accelerated coordinate system. In that way we can use the special relativistic laws of nature to understand the effect of a gravitational field.

As we have seen, the relativistic analogue of a uniformly accelerated frame is Rindler space. Because Rindler space covers only a portion of the space-time geometry (Region I) there are new and subtle features to the description of quantum fields. These features are closely associated with the existence of the horizon. The method we will use applies to any relativistic quantum field theory including those with nontrivial interactions. For illustrative purposes we will consider a free scalar field theory. It is important to bare in mind that such a non-interacting description is of limited validity. As we shall see, interactions become very important near the horizon of a black hole. Ignoring them leads to an inconsistent description of the Hawking evaporation process.

3.1 Classical Fields

First let us consider the evolution of a classical field in Rindler space. The field in Region I of Figure 1.2 can be described in a self contained way. Obviously influences from Regions II and III can never be felt in Region I since no point in Regions II or III is in the causal past of any point in Region I. Signals from Region IV can, of course, reach Region I, but to do so they must pass through the surface $\omega = -\infty$. Therefore signals from Region IV are regarded as initial data in the remote past by the Rindler observer. Evidently the Rindler observer sees a world in which physical phenomena can be described in a completely self contained way.

The evolution from one surface of constant ω to another is governed by the Rindler Hamiltonian. Using conventional methods the generator of ω -translations is given by

$$H_R = \int_{\rho=0}^{\infty} d\rho dX dY \rho T^{00}(\rho, X, Y) \quad (3.1.1)$$

where T^{00} is the usual Hamiltonian density used by the Minkowski observer. For example, for a massive scalar field with potential V , T^{00} is given by

$$T^{00} = \frac{\Pi^2}{2} + \frac{1}{2} (\nabla\chi)^2 + V(\chi) \quad (3.1.2)$$

where Π is the canonical momentum conjugate to χ . The Rindler Hamiltonian is

$$H_R = \int d\rho dx_{\perp} \frac{\rho}{2} \left[\Pi^2 + \left(\frac{\partial\chi}{\partial\rho} \right)^2 + \left(\frac{\partial\chi}{\partial x_{\perp}} \right)^2 + 2V(\chi) \right] \quad (3.1.3)$$

The origin of the factor ρ in the Rindler Hamiltonian density is straightforward. In Figure 3.1 the relation between neighboring equal Rindler-time surfaces is shown. The proper time separation between the surfaces is

$$\delta\tau = \rho \delta\omega \quad (3.1.4)$$

Thus, to push the ω -surface ahead requires a ρ -dependent time translation. This is the reason that T^{00} is weighted with the factor ρ . The Rindler Hamiltonian is similar to the generator of Lorentz boosts from the viewpoint of the Minkowski observer. However it only involves the degrees of freedom in Region I.

3.2 Entanglement

Quantum fields can also be described in a self-contained fashion in Rindler space, but a new twist is encountered. Our goal is to describe the usual physics of a quantum field in Minkowski space, but from the viewpoint of the Fidus in Region I, i.e. in Rindler space. To understand the new feature, recall that in the usual vacuum state, the correlation between fields at different spatial points does not vanish. For example, in free massless scalar theory the equal time correlator is given by

$$\langle 0 | \chi(X, Y, Z) \chi(X', Y', Z') | 0 \rangle \sim \frac{1}{\Delta^2} \quad (3.2.5)$$

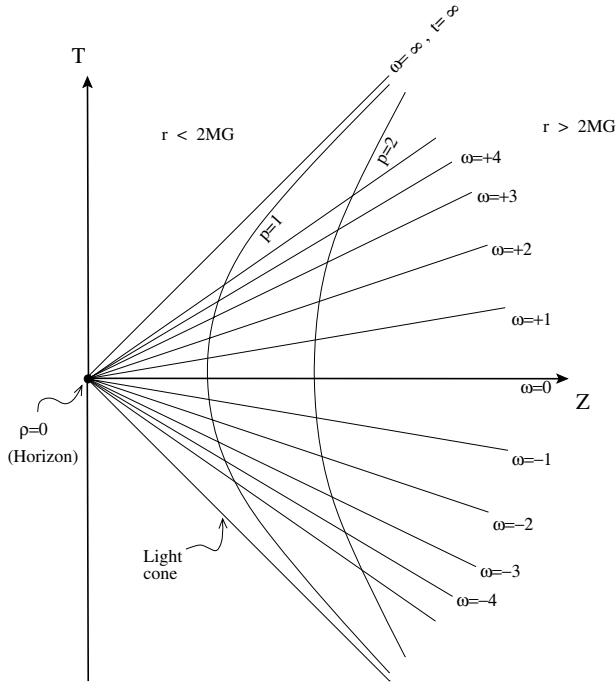


Fig. 3.1 *Equal time and proper distance surfaces in Rindler space*

where Δ is the space-like separation between the points (X, Y, Z) and (X', Y', Z')

$$\Delta^2 = (X - X')^2 + (Y - Y')^2 + (Z - Z')^2 \quad (3.2.6)$$

The two points might both lie within Region I, in which case the correlator in equation 3.2.5 represents the quantum correlation seen by Fido's in Region I. On the other hand, the two points might lie on opposite sides of the horizon at $Z = 0$. In that case the correlation is unmeasurable to the Fidos in Region I. Nevertheless it has significance. When two subsystems (fields in Regions I and III) become correlated, we say that they are quantum entangled, so that neither can be described in terms of pure states. The appropriate description of an entangled subsystem is in terms of a density matrix.

3.3 Review of the Density Matrix

Suppose a system consists of two subsystems, A and B , which have previously been in contact but are no longer interacting. The combined system has a wavefunction

$$\Psi = \Psi(\alpha, \beta) \quad (3.3.7)$$

where α and β are appropriate commuting variables for the subsystems A and B .

Now suppose we are only interested in subsystem A . A complete description of all measurements of A is provided by the density matrix $\rho_A(\alpha, \alpha')$.

$$\rho_A(\alpha, \alpha') = \sum_{\beta} \Psi^*(\alpha, \beta) \Psi(\alpha', \beta) \quad (3.3.8)$$

Similarly, experiments performed on B are described by $\rho_B(\beta, \beta')$.

$$\rho_B(\beta, \beta') = \sum_{\alpha} \Psi^*(\alpha, \beta) \Psi(\alpha, \beta') \quad (3.3.9)$$

The rule for computing an expectation value of an operator $\mathbf{a}$ composed of A degrees of freedom is

$$\langle \mathbf{a} \rangle = \text{Tr} \mathbf{a} \rho_A \quad (3.3.10)$$

Density matrices have the following properties:

- 1) $\text{Tr} \rho = 1$ (total probability=1)
- 2) $\rho = \rho^\dagger$ (hermiticity)
- 3) $\rho_j \geq 0$ (all eigenvalues are positive or zero)

In the representation in which ρ is diagonal

$$\rho = \begin{pmatrix} \rho_1 & 0 & 0 & \dots & \dots \\ 0 & \rho_2 & 0 & \dots & \dots \\ 0 & 0 & \rho_3 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \quad (3.3.11)$$

The eigenvalues ρ_j can be considered to be probabilities that the system is in the j^{th} state. However, unlike the case of a coherent superposition of states, the relative phases between the states $|j\rangle$ are random.

There is one special case when the density matrix is indistinguishable from a pure state. This is the case in which only one eigenvalue ρ_j is

nonzero. This case can only result from an uncorrelated product wave function of the form

$$\Psi(\alpha, \beta) = \psi_A(\alpha) \psi_B(\beta) \quad (3.3.12)$$

A quantitative measure of the departure from a pure state is provided by the *Von Neumann entropy*

$$S = -\text{Tr } \rho \log \rho = -\sum_j \rho_j \log \rho_j. \quad (3.3.13)$$

S is zero if and only if all the eigenvalues but one are zero. The one non-vanishing eigenvalue is equal to 1 by virtue of the trace condition on ρ . The entropy is also a measure of the degree of entanglement between A and B . It is therefore called the *entropy of entanglement*.

The opposite extreme to a pure state is a completely incoherent density matrix in which all the eigenvalues are equal to $\frac{1}{N}$, where N is the dimensionality of the Hilbert space. In that case S takes its maximum value

$$S_{\max} = -\sum_j \frac{1}{N} \log \frac{1}{N} = \log N \quad (3.3.14)$$

More generally, if ρ is a projection operator onto a subspace of dimension n , we find

$$S = \log n \quad (3.3.15)$$

Thus we see that the Von Neumann entropy is a measure of the number of states which have an appreciable probability in the statistical ensemble. We may think of e^S as an effective dimensionality of the subspace described by ρ .

The Von Neumann (or entanglement) entropy should not be confused with the thermal entropy of the second law of thermodynamics. This entropy has its origin in coarse graining. If a system with Hamiltonian H is in thermal equilibrium at temperature $T = 1/\beta$ then it is described by a Maxwell–Boltzman density matrix

$$\rho_{M.B.} = \frac{e^{-\beta H}}{\text{Tr } e^{-\beta H}}. \quad (3.3.16)$$

In this case the thermal entropy is given by

$$S_{\text{thermal}} = -\text{Tr } \rho_{M.B.} \log \rho_{M.B.} \quad (3.3.17)$$

3.4 The Unruh Density Matrix

Now let us consider the space of states describing a Lorentz invariant quantum field theory in Minkowski space. In Figure 3.2 the surface $T = 0$ of Minkowski space is shown divided into two halves, one in Region I and one in Region III. For the case of a scalar field χ the fields at each point

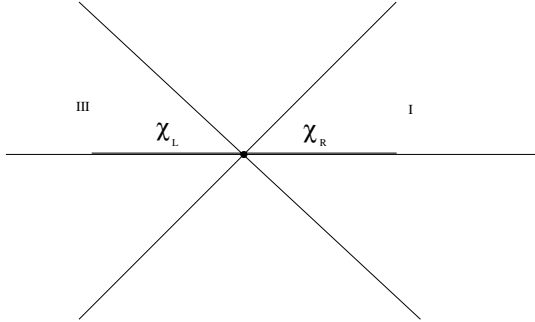


Fig. 3.2 *Fields on the spacelike surface $T = 0$ in Minkowski space*

of space form a complete set of commuting operators. These fields may be decomposed into two subsets associated with regions I and III. we call them χ_R and χ_L respectively. Thus

$$\chi(X, Y, Z) = \chi_R(X, Y, Z) \quad Z > 0 \quad (3.4.18)$$

$$\chi(X, Y, Z) = \chi_L(X, Y, Z) \quad Z < 0$$

The general wave functional of the system is a functional of χ_L and χ_R

$$\Psi = \Psi(\chi_L, \chi_R) \quad (3.4.19)$$

We wish to compute the density matrix used by the Fidos in Region I to describe their Rindler world. In particular we would like to understand the density matrix ρ_R which represents the usual Minkowski vacuum to the Fidos in Region I.

First let us see what we can learn from general principles. Obviously the state Ψ is translationally invariant under the usual Minkowski space translations. Thus the Fidos must see the vacuum as invariant under translations along the X and Y axes. However, the translation invariance along the Z axis is explicitly broken by the act of singling out the origin $Z = 0$ for

special consideration. From the X, Y translation invariance we conclude that ρ_R commutes with the components of momentum in these directions

$$[p_X, \rho_R] = [p_Y, \rho_R] = 0 \quad (3.4.20)$$

A very important property of ρ_R is that it is invariant under Rindler time translations $\omega \rightarrow \omega + \text{constant}$. This follows from the Lorentz boost invariance of Ψ . Thus

$$[H_R, \rho_R] = 0 \quad (3.4.21)$$

To proceed further we must use the fact that $\Psi(\chi_L, \chi_R)$ is the ground state of the Minkowski Hamiltonian. General path integral methods may be brought to bare on the computation of the ground state wave functional. Let us assume that the field theory is described in terms of an action

$$I = \int d^3X dT L \quad (3.4.22)$$

The so-called Euclidean field theory is defined by replacing the time coordinate T by iX^0 . For example, the Euclidean version of ordinary scalar field theory is obtained from the usual Minkowski action

$$I = \int d^3X dT \frac{1}{2} [\dot{\chi}^2 - (\nabla\chi)^2 - 2V(\chi)] \quad (3.4.23)$$

Letting $T \rightarrow iX^0$ we obtain the Euclidean action

$$I_E = \int d^4X \frac{1}{2} [(\partial_X\chi)^2 - 2V(\chi)] \quad (3.4.24)$$

Now a standard method of computing the ground state by path integration is to use the Feynman–Hellman theorem. Suppose we wish to compute $\Psi(\chi_L, \chi_R)$. Then we consider the path integral

$$\Psi(\chi_L, \chi_R) = \frac{1}{\sqrt{Z}} \int d\chi(x) e^{-I_E} \quad (3.4.25)$$

where the path integral is over all $\chi(x)$ with $X^0 > 0$ and Z is an appropriate normalization factor. The field $\chi(x)$ is constrained to equal (χ_L, χ_R) on the surface $X^0 = 0$. Finally the action I_E is evaluated as an integral over the portion of Minkowski space with $X^0 > 0$.

The boost invariance of the original Minkowski action insures that the Euclidean action has four dimensional rotation invariance. In particular, the invariance under ω -translations becomes invariance under rotations in the Euclidean (Z, X^0) plane. This suggests a new way to carry out the