



An Introduction to **String Theory** and **D-Brane Dynamics**

With Problems and Solutions

2nd Edition



Richard J Szabo

Imperial College Press



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Preface to Second Edition

The second edition of this book has improved in two main ways. Firstly, a few minor typos which plagued the first edition have been corrected. Secondly, and most notably, an extra chapter at the end has been added with detailed solutions to all exercises that appear in the main part of the text. These exercises are meant both to fill in some gaps in the main text, and also to provide the student reader with the rudimentary computational skills required in the field. The inclusion of this extra chapter of solutions should help make this second edition more complete and self-contained. It was decided to keep the overall style of the rest of the book intact, and hence the remaining chapters still serve as a brief, concise and quick introduction into the basic aspects of string theory and D-brane physics.

The author would like to thank Laurent Chaminade and others at Imperial College Press for the encouragement to produce this second edition. This work was supported in part by grant ST/G000514/1 “String Theory Scotland” from the UK Science and Technology Facilities Council.

Richard J. Szabo
Edinburgh, 2010

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Preface to First Edition

These notes comprise an expanded version of the string theory lectures given by the author at the 31st and 32nd British Universities Summer Schools on Theoretical Elementary Particle Physics (BUSSTEPP) which were held, respectively, in Manchester, England in 2001 and in Glasgow, Scotland in 2002, and also at the Pacific Institute for the Mathematical Sciences *Frontiers of Mathematical Physics* Summer School on “Strings, Gravity and Cosmology” which was held in Vancouver, Canada in 2003. The schools were attended mostly by Ph.D. students in theoretical high-energy physics who had just completed their first year of graduate studies. The lectures were thereby appropriately geared for this level. No prior knowledge of string theory was assumed, but a good background in quantum field theory, introductory level particle physics and group theory was expected. An acquaintance with the basic ideas of general relativity was helpful but not absolutely essential. Some familiarity with supersymmetry was also assumed because the supersymmetry lectures preceded the string theory lectures at the schools, although the full-blown machinery and techniques of supersymmetry were not exploited to any large extent.

The main references for string theory used during the courses were the standard books on the subject [Green, Schwarz and Witten (1987); Polchinski (1998)] and the more recent review article [Johnson (2001)]. The prerequisite supersymmetry lectures can be found in [Figueroa-O’Farrill (2001)]. Further references are cited in the text, but are mostly included for historical reasons and are by no means exhaustive. Complete sets of references may be found in the various cited books and review articles.

The lectures were delivered in the morning and exercises were assigned. These problems are also included in these notes. Many of them are intended to fill in the technical gaps which due to time constraints were not covered in

the lectures. Others are intended to give the student a better grasp of some “stringy” topics. This book has expanded on many aspects of string theory that were addressed during the schools, mainly to make the presentation clearer.

There were six one-hour lectures in total. Since string theory is nowadays such a vast and extensive subject, some focus on the subject material was of course required. The lectures differ perhaps from most introductory approaches since the intent was to provide the student not only with the rudiments of perturbative string theory, but also with an introduction to the more recently discovered non-perturbative degrees of freedom known as “D-branes”, which in the past few years have revolutionized the thinking about string theory and have brought the subject to the forefront of modern theoretical particle physics once again. This means that much of the standard introductory presentation was streamlined in order to allow for an introduction to these more current developments. The hope was that the student will have been provided with enough background to feel comfortable in starting to read current research articles, in addition to being exposed to some of the standard computational techniques in the field.

The basic perturbative material was covered in roughly the first three lectures and comprises chapters 1–4. Lecture 4 (chapter 5) then started to rapidly move towards explaining what D-branes are, and at the same time introducing some more novel stringy physics. Lectures 5 and 6 (chapters 6 and 7) then dealt with D-branes in detail, studied their dynamics, and provided a brief account of the gauge theory/string theory correspondence which has been such an active area of research over the past few years. For completeness, an extra chapter has also been added (chapter 8) which deals with the Ramond–Ramond couplings of D-branes and other novel aspects of D-brane dynamics such as the important “branes within branes” phenomenon. This final chapter is somewhat more advanced and is geared at the reader with some familiarity in differential topology and geometry.

The author is grateful to the participants, tutors and lecturers of the schools for their many questions, corrections, suggestions and criticisms which have all gone into the preparation of these lecture notes. He would especially like to thank M. Alford, C. Davies, J. Forshaw, M. Rozali and G. Semenoff for having organised these excellent schools, and for the encouragement to write up these notes. He would also like to thank J. Figueroa-O’Farrill and F. Lizzi for practical comments on the manuscript.

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Richard J. Szabo

Edinburgh, 2003

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Chapter 1

A Brief History of String Theory

To help introduce the topics which follow and their significance in high energy physics, in this chapter we will briefly give a non-technical historical account of the development of string theory to date, focusing on its achievements, failures and prospects. This will also help to motivate the vast interest in string theory within the particle theory community. It will further give an overview of the material which will follow.

In conventional quantum field theory, the fundamental objects are mathematical points in spacetime, modeling the elementary point particles of nature. String theory is a rather radical generalization of quantum field theory whereby the fundamental objects are extended one-dimensional lines or loops (Fig. 1.1). The various elementary particles observed in nature correspond to different vibrational modes of the string. While we cannot see a string (yet) in nature, if we are very far away from it we will be able to see its point-like oscillations, and hence measure the elementary particles that it produces. The main advantage of this description is that while there are many particles, there is only one string. This indicates that strings could serve as a good starting point for a unified field theory of the fundamental interactions.

This is the idea that emerged by the end of the 1960s from several years of intensive studies of dual models of hadron resonances [Veneziano (1968)]. In this setting, string theory attempts to describe the strong nuclear force. The excitement over this formalism arose from the fact that string S-matrix scattering amplitudes agreed with those found in meson scattering experiments at the time. The inclusion of fermions into the model led to the notion of a supersymmetric string, or “superstring” for short [Neveu and Schwarz (1971); Ramond (1971)]. The massive particles sit on “Regge trajectories” in this setting.

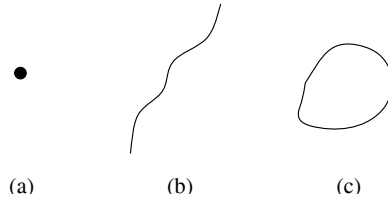


Fig. 1.1 (a) A point particle. (b) An open string. (c) A closed string.

However, around 1973 the interest in string theory quickly began to fade, mainly because quantum chromodynamics became recognized as the correct quantum field theory of the strong interactions. In addition, string theories possessed various undesirable features which made them inappropriate for a theory of hadrons. Among these were the large number of extra spacetime dimensions demanded by string theory, and the existence of massless particles other than the spin 1 gluon in the spectrum of string states.

In 1974 the interest in string theory was revived for another reason [Scherk and Schwarz (1974); Yoneya (1974)]. It was found that, among the massless string states, there is a spin 2 particle that interacts like a graviton. In fact, the only consistent interactions of massless spin 2 particles are gravitational interactions. Thus string theory naturally includes general relativity, and it was thereby proposed as a unified theory of the fundamental forces of nature, including gravity, rather than a theory of hadrons. This situation is in marked contrast to that in ordinary quantum field theory, which does not allow gravity to exist because its scattering amplitudes that involve graviton exchanges are severely plagued by non-renormalizable ultraviolet divergences (Fig. 1.2). On the other hand, string theory is a consistent quantum theory, free from ultraviolet divergences, which necessarily *requires* gravitation for its overall consistency.

With these facts it is possible to estimate the energy or length scale at which strings should be observed in nature. Since string theory is a relativistic quantum theory that includes gravity, it must involve the corresponding three fundamental constants, namely the speed of light c , the reduced Planck constant $\hbar$, and the Newtonian gravitational constant G . These three constants may be combined into a constant with dimensions of length. The characteristic length scale of strings may thereby be estimated

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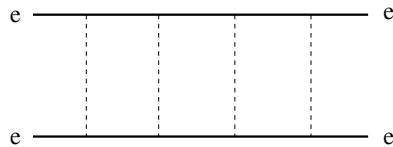


Fig. 1.2 A non-renormalizable ultraviolet divergent Feynman diagram in quantum gravity. The dashed lines depict graviton exchanges.

by the *Planck length* of quantum gravity:

$$\ell_P = \left( \frac{\hbar G}{c^3} \right)^{3/2} = 1.6 \times 10^{-33} \text{ cm} . \quad (1.1)$$

This is to be compared with the typical size of hadrons, which is of the order of  $10^{-13}$  cm. The corresponding energy scale is known as the *Planck mass*:

$$m_P = \left( \frac{\hbar c}{G} \right)^{1/2} = 1.2 \times 10^{19} \text{ GeV}/c^2 . \quad (1.2)$$

These scales indicate the reasons why strings have not been observed in nature thus far. Present day particle accelerators run at energies  $\ll m_P c^2$  and thus cannot resolve distances as short as the Planck length. At such energies, strings look like point particles, because at very large distance scales compared to the Planck length all one can observe is the string's center of mass motion, which is point-like. Thus at these present day scales, strings are accurately described by quantum field theory.

For many of the subsequent years superstring theory began showing great promise as a unified quantum theory of all the fundamental forces including gravity. Some of the general features which were discovered are:

- General relativity gets modified at very short distances/high energies (below the Planck scale), but at ordinary distances and energies it is present in string theory in exactly the same form as Einstein's theory.

- “Standard model type” Yang–Mills gauge theories arise very naturally in string theory. However, the reasons why the gauge group  $SU(3) \times SU(2) \times U(1)$  of the standard model should be singled out is not yet fully understood.
- String theory predicts supersymmetry, because its mathematical consistency depends crucially on it. This is a generic feature of string theory that has not yet been discovered experimentally.

This was the situation for some years, and again the interest in string theory within the high energy physics community began to fade. Different versions of superstring theory existed, but none of them resembled very closely the structure of the standard model.

Things took a sharp turn in 1985 with the birth of what is known as the “first superstring revolution”. The dramatic achievement at this time was the realization of how to cancel certain mathematical inconsistencies in quantum string theory. This is known as Green–Schwarz anomaly cancellation [Green and Schwarz (1984)] and its main consequence is that it leaves us with only five consistent superstring theories, each living in ten space-time dimensions. These five theories are called Type I, Type IIA, Type IIB,  $SO(32)$  heterotic, and  $E_8 \times E_8$  heterotic. The terminology will be explained later on. For now, we simply note the supersymmetric Yang–Mills gauge groups that arise in these theories. The Type I theories have gauge group  $SO(32)$ , both Type II theories have  $U(1)$ , and the heterotic theories have gauge groups as in their names. Of particular phenomenological interest was the  $E_8 \times E_8$  heterotic string, because from it one could construct grand unified field theories starting from the exceptional gauge group  $E_6$ .

The spacetime dimensionality problem is reconciled through the notion of “compactification”. Putting six of the spatial directions on a “small” six-dimensional compact space, smaller than the resolution of the most powerful microscope, makes the 9+1 dimensional spacetime look 3+1 dimensional, as in our observable world. The six dimensional manifolds are restricted by string dynamics to be “Calabi–Yau spaces” [Candelas *et al* (1985)]. These compactifications have tantalizingly similar features to the standard model. However, no complete quantitative agreement has been found yet between the two theories, such as the masses of the various elementary particles. This reason, and others, once again led to the demise of string theory towards the end of the 1980s. Furthermore, at that stage one only understood how to formulate superstring theories in terms of divergent perturbation series analogous to quantum field theory. Like in quantum chromodynamics, it

is unlikely that a realistic vacuum can be accurately analysed within perturbation theory. Without a good understanding of nonperturbative effects (such as the analogs of QCD instantons), superstring theory cannot give explicit, quantitative predictions for a grand unified model.

This was the state of affairs until around 1995 when the “second superstring revolution” set in. For the first time, it was understood how to go beyond the perturbation expansion of string theory via “dualities” which probe nonperturbative features of string theory [Font *et al* (1990); Hull and Townsend (1995); Kachru and Vafa (1995); Schwarz (1995); Sen (1994)]. The three major implications of these discoveries were:

- *Dualities relate all five superstring theories in ten dimensions to one another.*

The different theories are just perturbative expansions of a unique underlying theory  $\mathcal{U}$  about five different, consistent quantum vacua [Schwarz (1996); Schwarz (1997)]. Thus there is a completely unique theory of nature, whose equation of motion admits many vacua. This is of course a most desirable property of a unified theory.

- *The theory  $\mathcal{U}$  also has a solution called “M-Theory” which lives in 11 spacetime dimensions [Duff (1996); Townsend (1995); Witten (1995)].*

The low-energy limit of M-Theory is 11-dimensional supergravity [Cremmer, Julia and Scheck (1978)]. All five superstring theories can be thought of as originating from M-Theory [Duff (1996); Schwarz (1997)] (see Fig. 1.3). The underlying theory  $\mathcal{U}$  is depicted in Fig. 1.4.

- *In addition to the fundamental strings, the theory  $\mathcal{U}$  admits a variety of extended nonperturbative excitations called “ $p$ -branes”, where  $p$  is the number of spatial extensions of the objects [Horowitz and Strominger (1991)].*

Especially important in this regard are the “Dirichlet  $p$ -branes” [Dai, Leigh and Polchinski (1989); Hořava (1989); Polchinski (1995)], or “D-branes” for short, which are  $p$ -dimensional soliton-like hyperplanes in spacetime whose quantum dynamics are governed by the theory of *open strings* whose ends are constrained to move on them (Fig. 1.5).

We will not attempt any description of the theory  $\mathcal{U}$ , which at present is not very well understood. Rather, we wish to focus on the remarkable

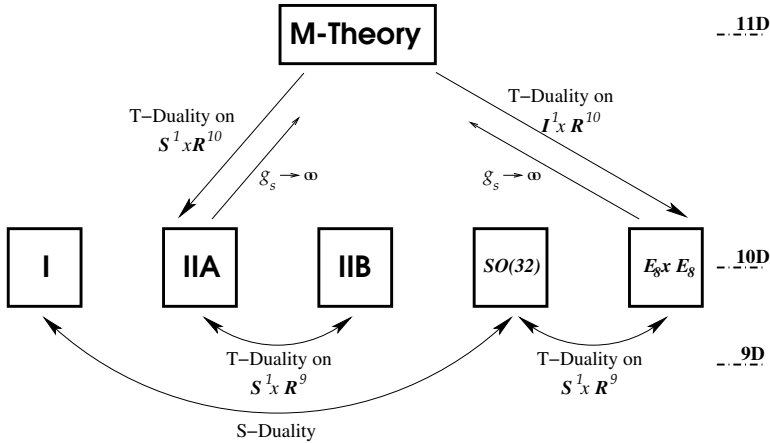


Fig. 1.3 The various duality transformations that relate the superstring theories in nine and ten dimensions. T-Duality inverts the radius  $R$  of the circle  $S^1$ , or the length of the finite interval  $I^1$ , along which a single direction of the spacetime is compactified, i.e.  $R \mapsto \ell_P^2/R$ . S-duality inverts the (dimensionless) string coupling constant  $g_s$ ,  $g_s \mapsto 1/g_s$ , and is the analog of electric-magnetic duality (or strong-weak coupling duality) in four-dimensional gauge theories. M-Theory originates as the strong coupling limit of either the Type IIA or  $E_8 \times E_8$  heterotic string theories.

impact in high-energy physics that the discovery of D-branes has provided. Amongst other things, they have led to:

- Explicit realizations of nonperturbative string dualities [Polchinski (1995)]. For example, an elementary closed string state in Theory A (which is perturbative because its amplitudes are functions of the string coupling  $g_s$ ) gets mapped under an S-duality transformation to a D-brane state in the dual Theory B (which depends on  $1/g_s$  and is therefore nonperturbative).
- A microscopic explanation of black hole entropy and the rate of emission of thermal (Hawking) radiation for black holes in string theory [Callan and Maldacena (1996); Strominger and Vafa (1996)].
- The gauge theory/gravity (or AdS/CFT) correspondence [Aharony *et al* (2000); Maldacena (1998)]. D-branes carry gauge fields, while on the other hand they admit a dual description as solutions of the classical equations of motion of string theory and supergravity. Demanding that these two descriptions be equivalent implies, for

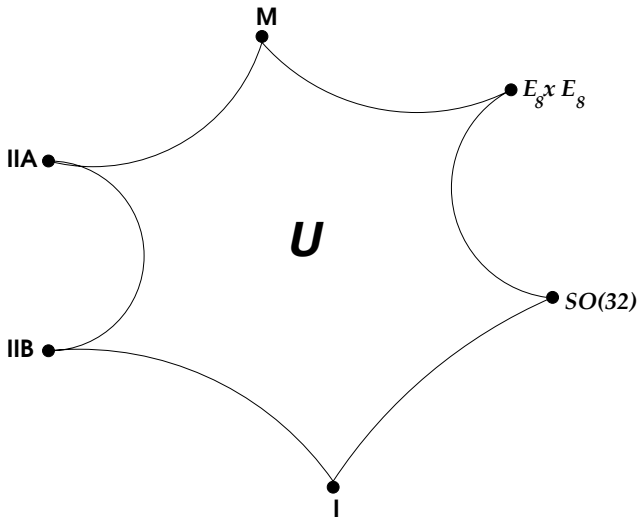


Fig. 1.4 The space  $\mathcal{U}$  of quantum string vacua. At each node a weakly-coupled string description is possible.

some special cases, that string theory is equivalent to a gauge field theory. This is an explicit realization of the old ideas that Yang–Mills theory may be represented as some sort of string theory.

- Probes of short-distances in spacetime [Douglas *et al* (1997)], where quantum gravitational fluctuations become important and classical general relativity breaks down.
- Large radius compactifications, whereby extra compact dimensions of size  $\gg (\text{TeV})^{-1}$  occur [Antoniadis *et al* (1998); Arkani-Hamed, Dimopoulos and Dvali (1998)]. This is the distance scale probed in present-day accelerator experiments, which has led to the hope that the extra dimensions required by string theory may actually be observable.
- Brane world scenarios, in which we model our world as a D-brane [Randall and Sundrum (1999a); Randall and Sundrum (1999b)]. This may be used to explain why gravity couples so weakly to matter, i.e. why the effective Planck mass in our 3+1-dimensional world is so large, and hence gives a potential explanation of the hierarchy problem  $m_P \gg m_{\text{weak}}$ .

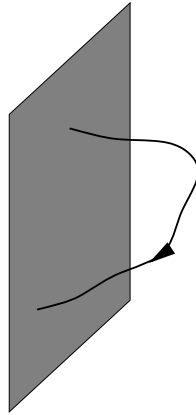


Fig. 1.5 A fundamental open string (wavy line) starting and ending (with Dirichlet boundary conditions) on a  $Dp$ -brane (shaded region) which is a  $p + 1$ -dimensional hyperplane in spacetime.

In what follows we will give the necessary background into the description of D-brane dynamics which leads to these exciting developments in theoretical high-energy physics.

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# Chapter 2

## Classical String Theory

In this chapter we will start making the notions of the previous chapter quantitative. We will treat the classical dynamics of strings, before moving on in the next chapter to the quantum theory. Usually, at least at introductory levels, one is introduced to quantum field theory by using “second quantization” which is based on field operators that create or destroy quanta. Here we will describe string dynamics in “first quantization” instead, which is a sum-over-histories or Feynman path integral approach closely tied to perturbation theory. To familiarize ourselves with this geometrical description, we will start by explaining how it can be used to reproduce the well-known dynamics of a massive, relativistic point particle. This will easily introduce the technique that will readily generalize to the case of extended objects such as strings and D-branes. We will then describe the bosonic string within this approach and analyse its classical equations of motion. For the remainder of this book we will work in natural units whereby  $c = \hbar = G = 1$ .

### 2.1 The Relativistic Particle

Consider a particle which propagates in  $d$ -dimensional “target spacetime”  $\mathbb{R}^{1,d-1}$  with coordinates

$$(t, \vec{x}) = x^\mu = (x^0, x^1, \dots, x^{d-1}) . \quad (2.1)$$

It sweeps out a path  $x^\mu(\tau)$  in spacetime, called a “worldline” of the particle, which is parametrized by a proper time coordinate  $\tau \in \mathbb{R}$  (Fig. 2.1). The infinitesimal, Lorentz-invariant path length swept out by the particle is

$$dl = (-ds^2)^{1/2} = (-\eta_{\mu\nu} dx^\mu dx^\nu)^{1/2} \equiv (-dx^\mu dx_\mu)^{1/2} , \quad (2.2)$$

where  $l$  is the proper-time of the particle and

$$(\eta_{\mu\nu}) = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \quad (2.3)$$

is the flat Minkowski metric. The action for a particle of mass  $m$  is then given by the total length of the trajectory swept out by the particle in spacetime:

$$S[x] = -m \int dl(\tau) = -m \int d\tau \sqrt{-\dot{x}^\mu \dot{x}_\mu} \quad (2.4)$$

where

$$\dot{x}^\mu \equiv \frac{dx^\mu}{d\tau} . \quad (2.5)$$

The minima of this action determine the trajectories of the particle with the smallest path length, and therefore the solutions to the classical equations of motion are the geodesics of the free particle in spacetime.

**Exercise 2.1.** *Show that the Euler–Lagrange equations resulting from the action (2.4) give the usual equations of relativistic particle kinematics:*

$$\dot{p}^\nu = 0 , \quad p^\nu = \frac{m \dot{x}^\nu}{\sqrt{-\dot{x}^\mu \dot{x}_\mu}} .$$

### 2.1.1 Reparametrization Invariance

The Einstein constraint  $p^2 \equiv p^\mu p_\mu = -m^2$  on the classical trajectories of the particle (Exercise 2.1) is related to the fact that the action (2.4) is invariant under arbitrary, local reparametrizations of the worldline, i.e.

$$\tau \longmapsto \tau(\tau') , \quad \frac{d\tau}{d\tau'} > 0 . \quad (2.6)$$

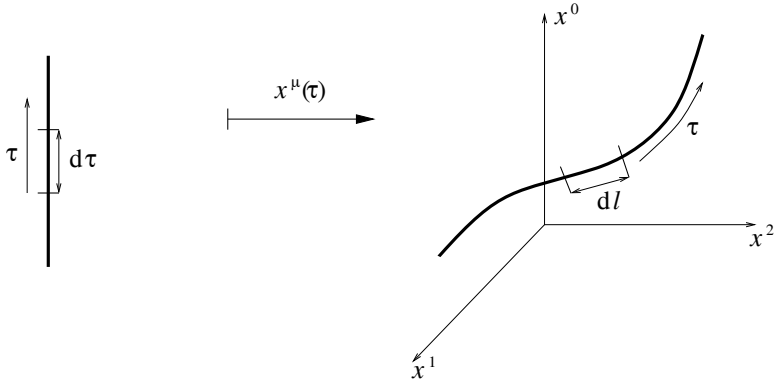


Fig. 2.1 The embedding  $\tau \mapsto x^\mu(\tau)$  of a particle trajectory into  $d$ -dimensional space-time. As  $\tau$  increases the particle propagates along its one-dimensional worldline in the target space.

This is a kind of local worldline “gauge invariance” which means that the form of  $S[x]$  is unchanged under such a coordinate change, since

$$d\tau \sqrt{-\dot{x}^\mu(\tau)\dot{x}_\mu(\tau)} = d\tau' \sqrt{-\dot{x}^\mu(\tau')\dot{x}_\mu(\tau')} . \quad (2.7)$$

It is a one-dimensional version of the usual four-dimensional general coordinate invariance in general relativity, in the sense that it corresponds to a worldline diffeomorphism symmetry of the theory. An application of the standard Noether procedure to the 0+1-dimensional field theory (2.4) leads to a conserved Noether current whose continuity equation is precisely the constraint  $p^2 = -m^2$ . It tells us how to eliminate one of the  $p^\mu$ ’s, and in the quantum theory it becomes the requirement that physical states and observables must be gauge invariant. It enables us to select a suitable gauge. Let us look at a couple of simple examples, as this point will be crucial for our later generalizations.

### 2.1.2 Examples

**Example 2.1.** The static gauge choice corresponds to taking

$$x^0 = \tau \equiv t \quad (2.8)$$

in which case the action assumes the simple form

$$S[x] = -m \int dt \sqrt{1 - v^2} \quad (2.9)$$

where

$$\vec{v} = \frac{d\vec{x}}{dt} \quad (2.10)$$

is the velocity of the particle. The equations of motion in this gauge take the standard form of those for a free, massive relativistic particle:

$$\frac{d\vec{p}}{dt} = \vec{0}, \quad \vec{p} = \frac{m \vec{v}}{\sqrt{1 - v^2}}. \quad (2.11)$$

**Example 2.2.** An even simpler gauge choice, known as the Galilean gauge, results from selecting

$$\dot{x}^\mu \dot{x}_\mu = -1. \quad (2.12)$$

The momentum of the particle in this gauge is given by the non-relativistic form (cf. Exercise 2.1)

$$p^\mu = m \dot{x}^\mu, \quad (2.13)$$

and the equations of motion are therefore

$$\ddot{x}^\mu = 0 \quad (2.14)$$

whose solutions are given by the Galilean trajectories

$$x^\mu(\tau) = x^\mu(0) + p^\mu \tau. \quad (2.15)$$

## 2.2 The Bosonic String

In the previous section we analysed a point particle, which is a zero-dimensional object described by a one-dimensional worldline in spacetime. We can easily generalize this construction to a *string*, which is a one-dimensional object described by a two-dimensional “worldsheet” that the string sweeps out as it moves in time with coordinates

$$(\xi^0, \xi^1) = (\tau, \sigma). \quad (2.16)$$

Here  $0 \leq \sigma \leq \pi$  is the spatial coordinate along the string, while  $\tau \in \mathbb{R}$  describes its propagation in time. The string’s evolution in time is described by functions  $x^\mu(\tau, \sigma)$ ,  $\mu = 0, 1, \dots, d-1$  giving the shape of its worldsheet

*Worldsheet diagram*

in the target spacetime (Fig. 2.2). The “induced metric”  $h_{ab}$  on the string worldsheet corresponding to its embedding into spacetime is given by the “pullback” of the flat Minkowski metric  $\eta_{\mu\nu}$  to the surface,

$$h_{ab} = \eta_{\mu\nu} \partial_a x^\mu \partial_b x^\nu , \quad (2.17)$$

where<sup>1</sup>

$$\partial_a \equiv \frac{\partial}{\partial \xi^a} , \quad a = 0, 1 . \quad (2.18)$$

An elementary calculation shows that the invariant, infinitesimal area element on the worldsheet is given by

$$dA = \sqrt{-\det(h_{ab})} d^2 \xi , \quad (2.19)$$

where the determinant is taken over the indices  $a, b = 0, 1$  of the  $2 \times 2$  symmetric nondegenerate matrix  $(h_{ab})$ .

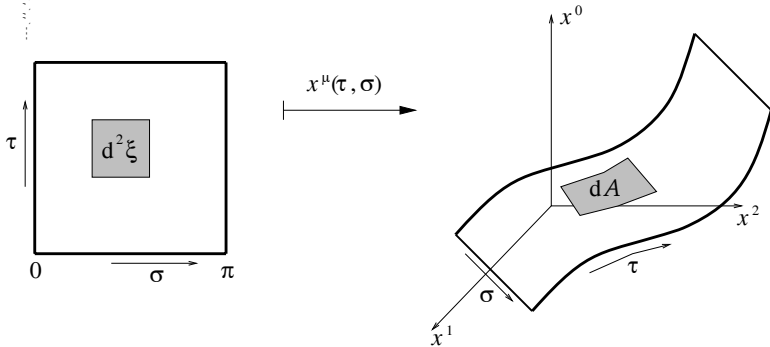


Fig. 2.2 The embedding  $(\tau, \sigma) \mapsto x^\mu(\tau, \sigma)$  of a string trajectory into  $d$ -dimensional spacetime. As  $\tau$  increases the string sweeps out its two-dimensional worldsheet in the target space, with  $\sigma$  giving the position along the string.

In analogy to the point particle case, we can now write down an action whose variational law minimizes the total area of the string worldsheet

<sup>1</sup>Notation: Greek letters  $\mu, \nu, \dots$  denote spacetime indices, beginning Latin letters  $a, b, \dots$  denote worldsheet indices, and later Latin letters  $i, j, \dots$  label spatial directions in the target space. Unless otherwise stated, we also adhere to the standard Einstein summation convention for summing over repeated upper and lower indices.

in spacetime:

$$S[x] = -T \int dA = -T \int d^2\xi \sqrt{-\det_{a,b} (\partial_a x^\mu \partial_b x_\mu)} . \quad (2.20)$$

The quantity  $T$  has dimensions of mass per unit length and is the *tension* of the string. It is related to the “intrinsic length”  $\ell_s$  of the string by

$$T = \frac{1}{2\pi\alpha'} , \quad \alpha' = \ell_s^2 . \quad (2.21)$$

The parameter  $\alpha'$  is called the “universal Regge slope”, because the string vibrational modes all lie on linear parallel Regge trajectories with slope  $\alpha'$ . The action (2.20) defines a 1+1-dimensional field theory on the string worldsheet with bosonic fields  $x^\mu(\tau, \sigma)$ .

**Exercise 2.2.** Show that the action (2.20) is reparametrization invariant, i.e. if  $\xi \mapsto \xi(\xi')$ , then it takes the same form when expressed in terms of the new worldsheet coordinates  $\xi'$ .

Evaluating the determinant explicitly in (2.20) leads to the form

$$S[x] = -T \int d\tau d\sigma \sqrt{\dot{x}^2 x'^2 - (\dot{x} \cdot x')^2} \quad (2.22)$$

where

$$\dot{x}^\mu = \frac{\partial x^\mu}{\partial \tau} , \quad x'^\mu = \frac{\partial x^\mu}{\partial \sigma} . \quad (2.23)$$

This is the form that the original string action appeared in and is known as the “Nambu–Goto action” [Goto (1971); Nambu (1974)]. However, the square root structure of this action is somewhat awkward to work with. It can, however, be eliminated by the fundamental observation that the

Nambu–Goto action is classically equivalent to another action which does not have the square root:

$$\begin{aligned}
 S[x, \gamma] &= -\frac{T}{2} \int d^2\xi \sqrt{-\gamma} \gamma^{ab} h_{ab} \\
 &= -\frac{T}{2} \int d^2\xi \sqrt{-\gamma} \gamma^{ab} \partial_a x^\mu \partial_b x^\nu \eta_{\mu\nu} .
 \end{aligned}
 \tag{2.24}$$

Here the auxilliary rank two symmetric tensor field  $\gamma_{ab}(\tau, \sigma)$  has a natural interpretation as a metric on the string worldsheet, and we have defined

$$\gamma = \det_{a,b} (\gamma_{ab}) , \quad \gamma^{ab} = (\gamma^{-1})^{ab} . \tag{2.25}$$

The action (2.24) is called the “Polyakov action” [Polyakov (1981a)].

**Exercise 2.3.** Show that the Euler-Lagrange equations obtained by varying the Polyakov action with respect to  $\gamma^{ab}$  are

$$T_{ab} \equiv \partial_a x^\mu \partial_b x_\mu - \frac{1}{2} \gamma_{ab} \gamma^{cd} \partial_c x^\mu \partial_d x_\mu = 0 , \quad a, b = 0, 1 .$$

Then show that this equation can be used to eliminate the worldsheet metric  $\gamma_{ab}$  from the action, and as such recovers the Nambu–Goto action.

### 2.2.1 Worldsheet Symmetries

The quantity  $T_{ab}$  appearing in Exercise 2.3 is the energy–momentum tensor of the 1+1-dimensional worldsheet field theory defined by the action (2.24). The conditions  $T_{ab} = 0$  are often referred to as “Virasoro constraints” [Virasoro (1970)] and they are equivalent to two local “gauge symmetries” of the Polyakov action, namely the “reparametrization invariance”

$$(\tau, \sigma) \longmapsto \left( \tau(\tau', \sigma'), \sigma(\tau', \sigma') \right) , \tag{2.26}$$

and the “Weyl invariance” (or “conformal invariance”)

$$\gamma_{ab} \longmapsto e^{2\rho(\tau, \sigma)} \gamma_{ab} \tag{2.27}$$

where  $\rho(\tau, \sigma)$  is an arbitrary function on the worldsheet. These two local symmetries of  $S[x, \gamma]$  allow us to select a gauge in which the three functions

residing in the symmetric  $2 \times 2$  matrix  $(\gamma_{ab})$  are expressed in terms of just a *single* function. A particularly convenient choice is the “conformal gauge”

$$(\gamma_{ab}) = e^{\phi(\tau,\sigma)} (\eta_{ab}) = e^{\phi(\tau,\sigma)} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (2.28)$$

In this gauge, the metric  $\gamma_{ab}$  is said to be “conformally flat”, because it agrees with the Minkowski metric  $\eta_{ab}$  of a flat worldsheet, but only up to the scaling function  $e^{\phi}$ . Then, at the classical level, the conformal factor  $e^{\phi}$  drops out of everything and we are left with the simple gauge-fixed action  $S[x, e^{\phi} \eta]$ , i.e. the Polyakov action in the conformal gauge, and the constraints  $T_{ab} = 0$ :

$$\begin{aligned} S[x, e^{\phi} \eta] &= T \int d\tau d\sigma (\dot{x}^2 - x'^2), \\ T_{01} = T_{10} &= \dot{x} \cdot x' = 0, \\ T_{00} = T_{11} &= \frac{1}{2} (\dot{x}^2 + x'^2) = 0. \end{aligned} \quad (2.29)$$

Note that apart from the constraints, (2.29) defines a *free* (noninteracting) field theory.

## 2.3 String Equations of Motion

The equations of motion for the bosonic string can be derived by applying the variational principle to the 1+1-dimensional field theory (2.29). Varying the Polyakov action in the conformal gauge with respect to the  $x^{\mu}$  gives

$$\delta S[x, e^{\phi} \eta] = T \int d\tau d\sigma (\eta^{ab} \partial_a \partial_b x_{\mu}) \delta x^{\mu} - T \int d\tau x'_{\mu} \delta x^{\mu} \Big|_{\sigma=0}^{\sigma=\pi}. \quad (2.30)$$

The first term in (2.30) yields the usual bulk equations of motion which here correspond to the two-dimensional wave equation

$$\left( \frac{\partial^2}{\partial \sigma^2} - \frac{\partial^2}{\partial \tau^2} \right) x^{\mu}(\tau, \sigma) = 0. \quad (2.31)$$

The second term comes from the integration by parts required to arrive at the bulk differential equation, which involves a total derivative over the spatial interval  $0 \leq \sigma \leq \pi$  of the string. In order that the total variation of the action be zero, these boundary terms must vanish as well. The manner

in which we choose them to vanish depends crucially on whether we are dealing with *closed* or *open* strings. The solutions of the classical equations of motion then correspond to solutions of the wave equation (2.31) with the appropriate boundary conditions.

Closed Strings : Here we tie the two ends of the string at  $\sigma = 0$  and  $\sigma = \pi$  together by imposing periodic boundary conditions on the string embedding fields (Fig. 2.3):

$$\begin{aligned} x^\mu(\tau, 0) &= x^\mu(\tau, \pi) , \\ x'^\mu(\tau, 0) &= x'^\mu(\tau, \pi) . \end{aligned} \quad (2.32)$$

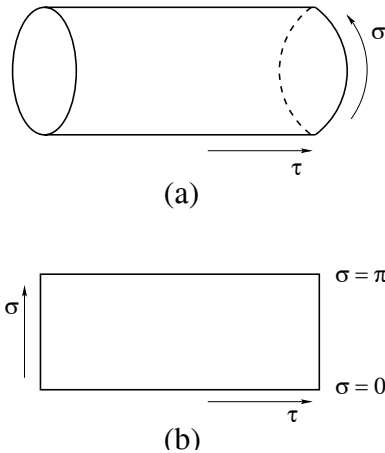


Fig. 2.3 The worldsheet of (a) a closed string is an infinite cylinder  $\mathbb{R} \times \mathbf{S}^1$ , and of (b) an open string is an infinite strip  $\mathbb{R} \times \mathbf{I}^1$ .

Open Strings : Here there are two canonical choices of boundary conditions.<sup>2</sup> *Neumann* boundary conditions are defined by

$$x'^\mu(\tau, \sigma) \Big|_{\sigma=0, \pi} = 0 . \quad (2.33)$$

In this case the ends of the string can sit anywhere in spacetime. *Dirichlet* boundary conditions, on the other hand, are defined by

$$\dot{x}^\mu(\tau, \sigma) \Big|_{\sigma=0, \pi} = 0 . \quad (2.34)$$

<sup>2</sup>These are by no means the only boundary conditions possible, but they are the only ones of direct physical relevance to the topics covered in this book.

Integrating the condition (2.34) over  $\tau$  specifies a spacetime location on which the string ends, and so Dirichlet boundary conditions are equivalent to fixing the endpoints of the string:

$$\delta x^\mu(\tau, \sigma) \Big|_{\sigma=0, \pi} = 0 . \quad (2.35)$$

We will see later on that this spacetime point corresponds to a physical object called a “D-brane”. For the time being, however, we shall focus our attention on Neumann boundary conditions for open strings.

### 2.3.1 Mode Expansions

To solve the equations of motion, we write the two-dimensional wave equation (2.31) in terms of worldsheet light-cone coordinates:

$$\partial_+ \partial_- x^\mu = 0 , \quad (2.36)$$

where

$$\xi^\pm = \tau \pm \sigma , \quad \partial_\pm = \frac{\partial}{\partial \xi^\pm} . \quad (2.37)$$

The general solution of (2.36) is then the sum of an analytic function of  $\xi^+$  alone, which we will call the “left-moving” solution, and an analytic function of  $\xi^-$  alone, which we call the “right-moving” solution,  $x^\mu(\tau, \sigma) = x_L^\mu(\xi^+) + x_R^\mu(\xi^-)$ . The precise form of the solutions now depends on the type of boundary conditions.

Closed Strings: The periodic boundary conditions (2.32) accordingly restrict the Taylor series expansions of the analytic functions which solve (2.36), and we arrive at the solution

$$\begin{aligned} x^\mu(\tau, \sigma) &= x_L^\mu(\xi^+) + x_R^\mu(\xi^-) , \\ x_L^\mu(\xi^+) &= \frac{1}{2} x_0^\mu + \alpha' p_0^\mu \xi^+ + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\tilde{\alpha}_n^\mu}{n} e^{-2i n \xi^+} , \\ x_R^\mu(\xi^-) &= \frac{1}{2} x_0^\mu + \alpha' p_0^\mu \xi^- + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\alpha_n^\mu}{n} e^{-2i n \xi^-} . \end{aligned} \quad (2.38)$$

We have appropriately normalized the terms in these Fourier-type series expansions, which we will refer to as “mode expansions”, according to physical dimension. Reality of the string embedding function  $x^\mu$  requires the integration constants  $x_0^\mu$  and  $p_0^\mu$  to be real, and

$$(\tilde{\alpha}_n^\mu)^* = \tilde{\alpha}_{-n}^\mu, \quad (\alpha_n^\mu)^* = \alpha_{-n}^\mu. \quad (2.39)$$

By integrating  $x^\mu$  and  $\dot{x}^\mu$  over  $\sigma \in [0, \pi]$  we see that  $x_0^\mu$  and  $p_0^\mu$  represent the center of mass position and momentum of the string, respectively. The  $\tilde{\alpha}_n^\mu$  and  $\alpha_n^\mu$  represent the oscillatory modes of the string. The mode expansions (2.38) correspond to those of left and right moving travelling waves circulating around the string in opposite directions.

**Open Strings :** For open strings, the spatial worldsheet coordinate  $\sigma$  lives on a finite interval rather than a circle. The open string mode expansion may be obtained from that of the closed string through the “doubling trick”, which identifies  $\sigma \sim -\sigma$  on the circle  $\mathbf{S}^1$  and thereby maps it onto the finite interval  $[0, \pi]$  (Fig. 2.4). The open string solution to the equations of motion may thereby be obtained from (2.38) by imposing the extra condition  $x^\mu(\tau, \sigma) = x^\mu(\tau, -\sigma)$ . This is of course still compatible with the wave equation (2.31) and it immediately implies the Neumann boundary conditions (2.33). We therefore find

$$x^\mu(\tau, \sigma) = x_0^\mu + 2\alpha' p_0^\mu \tau + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{\alpha_n^\mu}{n} e^{-in\tau} \cos(n\sigma). \quad (2.40)$$

The open string mode expansion has a standing wave for its solution, representing the left and right moving sectors reflected into one another by the Neumann boundary condition (2.33).

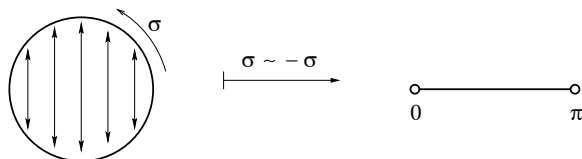


Fig. 2.4 The doubling trick identifies opposite points on the circle and maps it onto a finite interval.

### 2.3.2 Mass-Shell Constraints

The final ingredients to go into the classical solution are the physical constraints  $T_{ab} = 0$ . In the light-cone coordinates (2.37), the components  $T_{-+}$  and  $T_{+-}$  are identically zero, while in the case of the closed string the remaining components are given by

$$\begin{aligned} T_{++}(\xi^+) &= \frac{1}{2} \left( \partial_+ x_L \right)^2 = \sum_{n=-\infty}^{\infty} \tilde{L}_n e^{2in\xi^+} = 0, \\ T_{--}(\xi^-) &= \frac{1}{2} \left( \partial_- x_R \right)^2 = \sum_{n=-\infty}^{\infty} L_n e^{2in\xi^-} = 0, \end{aligned} \quad (2.41)$$

where we have defined

$$\tilde{L}_n = \frac{1}{2} \sum_{m=-\infty}^{\infty} \tilde{\alpha}_{n-m} \cdot \tilde{\alpha}_m, \quad L_n = \frac{1}{2} \sum_{m=-\infty}^{\infty} \alpha_{n-m} \cdot \alpha_m \quad (2.42)$$

with

$$\tilde{\alpha}_0^\mu = \alpha_0^\mu = \sqrt{\frac{\alpha'}{2}} p_0^\mu. \quad (2.43)$$

For open strings, we have only the constraint involving untilded quantities, and the definition of the zero modes (2.43) changes to  $\alpha_0^\mu = \sqrt{2\alpha'} p_0^\mu$ .

This gives an infinite number of constraints corresponding to an infinite number of conserved currents of the 1+1-dimensional field theory. They are associated with the residual, local, infinite-dimensional “conformal symmetry” of the theory which preserves the conformal gauge condition (2.28):

$$\begin{aligned} \xi^+ &\mapsto \xi'^+ = f(\xi^+), \\ \xi^- &\mapsto \xi'^- = g(\xi^-), \end{aligned} \quad (2.44)$$

where  $f$  and  $g$  are arbitrary analytic functions. Only the conformal factor  $\phi$  in (2.28) is affected by such coordinate transformations, and so the entire classical theory is invariant under them. They are known as “conformal transformations” and they rescale the induced worldsheet metric while leaving preserved all angles in two-dimensions. This “conformal invariance” of the worldsheet field theory makes it a “conformal field theory” [Belavin, Polyakov and Zamolodchikov (1984); Ginsparg (1990)], and represents one of the most powerful results and techniques of perturbative string theory.

## Chapter 3

# Quantization of the Bosonic String

We will now proceed to quantize the classical theory of the previous chapter. We will proceed by applying the standard canonical quantization techniques of quantum field theory to the 1+1-dimensional worldsheet field theory. Imposing the mass-shell constraints will thereby lead to the construction of physical states and the appearance of elementary particles in the quantum string spectrum. We will then turn to some more formal aspects of the quantum theory, namely a description of the operators which create particles from the string vacuum, a heuristic explanation of the structure of the string perturbation expansion, and how to incorporate non-abelian gauge degrees of freedom into the spectrum of string states.

### 3.1 Canonical Quantization

Recall from the previous chapter that the string worldsheet field theory is essentially a free field theory, and hence its quantization is particularly simple. The basic set-up for canonical quantization follows the standard practise of quantum field theory and is left as an exercise.

**Exercise 3.1.** (a) *Starting from the Polyakov action in the conformal gauge, show that the total classical Hamiltonian is given by*

$$H = \begin{cases} \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{-n} \cdot \alpha_n = L_0 & \text{for open strings ,} \\ \frac{1}{2} \sum_{n=-\infty}^{\infty} (\tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n + \alpha_{-n} \cdot \alpha_n) = \tilde{L}_0 + L_0 & \text{for closed strings .} \end{cases}$$

(b) Calculate the canonical momentum conjugate to the string embedding field  $x^\mu$ , and hence show that the oscillator modes have the quantum commutators

$$\begin{aligned} [x_0^\mu, p_0^\nu] &= i \eta^{\mu\nu} , \\ [x_0^\mu, x_0^\nu] &= [p_0^\mu, p_0^\nu] = 0 , \\ [\alpha_m^\mu, \alpha_n^\nu] &= [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m \delta_{m+n,0} \eta^{\mu\nu} , \\ [\alpha_m^\mu, \tilde{\alpha}_n^\nu] &= 0 . \end{aligned}$$

From Exercise 3.1 it follows that  $(a_m^\mu, a_m^{\mu\dagger}) = (\frac{1}{\sqrt{m}} \alpha_m^\mu, \frac{1}{\sqrt{m}} \alpha_{-m}^\mu)$  define quantum mechanical raising and lowering operators for the simple harmonic oscillator, i.e.

$$[a_m^\mu, a_m^{\mu\dagger}] = 1 . \quad (3.1)$$

This structure is of course anticipated, because free quantum fields are simply composed of infinitely many harmonic oscillators. The corresponding Hilbert space is therefore a Fock space spanned by products of states  $|n\rangle$ ,  $n = 0, 1, 2, \dots$ , which are built on a normalized ground state  $|0\rangle$  annihilated by the lowering operators  $a = a_m^\mu$ :

$$\begin{aligned} \langle 0|0\rangle &= 1 , \\ a|0\rangle &= 0 , \\ |n\rangle &= \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle . \end{aligned} \quad (3.2)$$

By repeated use of the commutation relation (3.1), in the usual way one can arrive at the relations

$$\begin{aligned} \langle m|n\rangle &= \delta_{nm} , \\ a^\dagger a|n\rangle &= n|n\rangle . \end{aligned} \quad (3.3)$$

In each sector of the theory, i.e. open, and closed left-moving and right-moving, we get  $d$  independent families of such infinite sets of oscillators, one for each spacetime dimension  $\mu = 0, 1, \dots, d-1$ . The only subtlety in this case is that, because  $\eta^{00} = -1$ , the time components are proportional to oscillators with the wrong sign, i.e.  $[a_m^0, a_n^{0\dagger}] = -1$ . Such oscillators are potentially dangerous, because they create states of negative norm which can lead to an inconsistent, non-unitary quantum theory (with negative probabilities and the like). However, as we will see, the Virasoro constraints

$T_{ab} = 0$  eliminate the negative norm states from the physical spectrum of the string.

From Exercise 3.1 it also follows that the zero mode operators  $x_0^\mu$  and  $p_0^\mu$  obey the standard Heisenberg commutation relations. They may thereby be represented on the Hilbert space spanned by the usual plane wave basis  $|k\rangle = e^{i k \cdot x}$  of eigenstates of  $p_0^\mu$ . Thus the Hilbert space of the string is built on the states  $|k; 0\rangle$  of center of mass momentum  $k^\mu$  with

$$\begin{aligned} p_0^\mu |k; 0\rangle &= k^\mu |k; 0\rangle, \\ \alpha_m^\mu |k; 0\rangle &= 0 \end{aligned} \quad (3.4)$$

for  $m > 0$  and  $\mu = 0, 1, \dots, d-1$ . For closed strings, there is also an independent left-moving copy of the Fock space.

### 3.1.1 Normal Ordering

For the quantum versions of the Virasoro operators defined in (2.42), we use the usual “normal ordering” prescription that places all lowering operators to the right. Then the  $L_n$  for  $n \neq 0$  are all fine when promoted to quantum operators. However, the Hamiltonian  $L_0$  needs more careful definition, because  $\alpha_n^\mu$  and  $\alpha_{-n}^\mu$  do not commute. As a quantum operator we have

$$L_0 = \frac{1}{2} \alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n + \varepsilon_0, \quad (3.5)$$

where

$$\varepsilon_0 = \frac{d-2}{2} \sum_{n=1}^{\infty} n \quad (3.6)$$

is the total zero-point energy of the families of infinite field oscillators (coming from the usual zero point energy  $\frac{1}{2}$  of the quantum mechanical harmonic oscillator). The factor of  $d-2$  appears, rather than  $d$ , because after imposition of the physical constraints there remain only  $d-2$  independent polarizations of the string embedding fields  $x^\mu$ . Explicitly, the constraints can be satisfied by imposing a spacetime light-cone gauge condition  $p^+ = 0$  which retains only transverse degrees of freedom.

The quantity  $\varepsilon_0$  is simply the Casimir energy arising from the fact that the 1+1-dimensional quantum field theory here is defined on a box, which is an infinite strip for open strings and an infinite cylinder for closed strings. Of course  $\varepsilon_0 = \infty$  formally, but, as usual in quantum field theory, it can

be regulated to give a finite answer corresponding to the total zero-point energy of all harmonic oscillators in the system. For this, we write it as

$$\varepsilon_0 = \frac{d-2}{2} \zeta(-1) , \quad (3.7)$$

where

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \quad (3.8)$$

for  $z \in \mathbb{C}$  is called the “Riemann zeta-function”. The function  $\zeta(z)$  can be regulated and has a well-defined analytical continuation to a finite function for  $\text{Re}(z) \leq 1$ . In this paper we will only require its values at  $z = 0$  and  $z = -1$  [Gradshteyn and Ryzhik (1980)]:

$$\begin{aligned} \zeta(0) &= -\frac{1}{2} , \\ \zeta(-1) &= -\frac{1}{12} . \end{aligned} \quad (3.9)$$

The vacuum energy is thereby found to be

$$\boxed{\varepsilon_0 = -\frac{d-2}{24} .} \quad (3.10)$$

**Exercise 3.2.** (a) Show that the angular momentum operators of the worldsheet field theory are given by

$$J^{\mu\nu} = x_0^\mu p_0^\nu - x_0^\nu p_0^\mu - i \sum_{n=1}^{\infty} \frac{1}{n} \left( \alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu \right) .$$

(b) Use the canonical commutation relations to verify that the Poincaré algebra

$$\begin{aligned} [p_0^\mu, p_0^\nu] &= 0 , \\ [p_0^\mu, J^{\nu\rho}] &= -i \eta^{\mu\nu} p_0^\rho + i \eta^{\mu\rho} p_0^\nu , \\ [J^{\mu\nu}, J^{\rho\lambda}] &= -i \eta^{\nu\rho} J^{\mu\lambda} + i \eta^{\mu\rho} J^{\nu\lambda} + i \eta^{\nu\lambda} J^{\mu\rho} - i \eta^{\mu\lambda} J^{\nu\rho} \end{aligned}$$

is satisfied.

(c) Show that for all  $n$ ,

$$[L_n, J^{\mu\nu}] = 0 .$$

This will guarantee later on that the string states are Lorentz multiplets.

~~~~~

3.2 The Physical String Spectrum

Our construction of quantum states of the bosonic string will rely heavily on a fundamental result that is at the heart of the conformal symmetry described at the end of section 2.3.

Exercise 3.3. *Use the oscillator algebra to show that the operators L_n generate the infinite-dimensional “Virasoro algebra of central charge $c = d$ ” [Virasoro (1970)],*

$$[L_n, L_m] = (n - m) L_{n+m} + \frac{c}{12} (n^3 - n) \delta_{n+m, 0} .$$

The constant term on the right-hand side of these commutation relations is often called the “conformal anomaly”, as it represents a quantum breaking of the classical conformal symmetry algebra [Polyakov (1981a)].

We define the “physical states” $|\text{phys}\rangle$ of the full Hilbert space to be those which obey the Virasoro constraints $T_{ab} \equiv 0$:

$$\begin{aligned} (L_0 - a)|\text{phys}\rangle &= 0 , \quad a \equiv -\varepsilon_0 > 0 , \\ L_n|\text{phys}\rangle &= 0 \end{aligned} \tag{3.11}$$

for $n > 0$. These constraints are just the analogs of the “Gupta–Bleuler prescription” for imposing mass-shell constraints in quantum electrodynamics. The L_0 constraint in (3.11) is a generalization of the Klein–Gordon equation, as it contains $p_0^2 = -\partial_\mu \partial^\mu$ plus oscillator terms. Note that because of the central term in the Virasoro algebra, it is inconsistent to impose these constraints on both L_n and L_{-n} .

3.2.1 The Open String Spectrum

We will begin by considering open strings as they are somewhat easier to describe. Mathematically, their spectrum is the same as that of the right-moving sector of closed strings. The closed string spectrum will thereby be straightforward to obtain afterwards. The constraint $L_0 = a$ in (3.11) is then equivalent to the “mass-shell condition”

$$m^2 = -p_0^2 = -\frac{1}{2\alpha'} \alpha_0^2 = \frac{1}{\alpha'} (N - a) ,$$

(3.12)

where N is the “level number” which is defined to be the oscillator number operator

$$N = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n = \sum_{n=1}^{\infty} n a_n^\dagger \cdot a_n = 0, 1, 2, \dots, \quad (3.13)$$

and $N_n \equiv a_n^\dagger \cdot a_n = 0, 1, 2, \dots$ is the usual number operator associated with the oscillator algebra (3.1).

Ground State $N = 0$: The ground state has a unique realization whereby all oscillators are in the Fock vacuum, and is therefore given by $|k; 0\rangle$. The momentum k of this state is constrained by the Virasoro constraints to have mass-squared given by

$$-k^2 = m^2 = -\frac{a}{\alpha'} < 0. \quad (3.14)$$

Since the vector k^μ is space-like, this state therefore describes a “tachyon”, i.e. a particle which travels faster than the speed of light. So the bosonic string theory is *not* a consistent quantum theory, because its vacuum has imaginary energy and hence is *unstable*. As in quantum field theory, the presence of a tachyon indicates that one is perturbing around a local maximum of the potential energy, and we are sitting in the wrong vacuum. However, in perturbation theory, which is the framework in which we are implicitly working here, this instability is not visible. Since we will eventually remedy the situation by studying tachyon-free superstring theories, let us just plug along without worrying for now about the tachyonic state.

First Excited Level $N = 1$: The only way to get $N = 1$ is to excite the first oscillator modes once, $\alpha_{-1}^\mu |k; 0\rangle$. We are also free to specify a “polarization vector” ζ_μ for the state. So the most general level 1 state is given by

$$|k; \zeta\rangle = \zeta \cdot \alpha_{-1} |k; 0\rangle. \quad (3.15)$$

The Virasoro constraints give the energy

$$m^2 = \frac{1}{\alpha'} (1 - a). \quad (3.16)$$

Furthermore, using the commutation relations and (2.42) we may compute

$$L_1 |k; \zeta\rangle = \sqrt{2\alpha'} (k \cdot \alpha_1) (\zeta \cdot \alpha_{-1}) |k; 0\rangle = \sqrt{2\alpha'} (k \cdot \zeta) |k; 0\rangle, \quad (3.17)$$

and thus the physical state condition $L_1 |k; \zeta\rangle = 0$ implies that the polarization and momentum of the state must obey

$$k \cdot \zeta = 0. \quad (3.18)$$

The cases $a \neq 1$ turn out to be unphysical, as they contain tachyons and ghost states of negative norm. So we shall take $a = 1$, which upon comparison with (3.10) fixes the “bosonic critical dimension of spacetime”:

$$\boxed{d = 26} . \quad (3.19)$$

The condition (3.19) can be regarded as the requirement of cancellation of the conformal anomaly in the quantum theory [Polyakov (1981a)], obtained by demanding the equivalence of the quantizations in both the light-cone and conformal gauges. The latter approach relies on the introduction of worldsheet Faddeev–Popov ghost fields for the gauge-fixing of the conformal gauge in the quantum theory.

Exercise 3.4. Consider the “spurious state” defined by $|\psi\rangle = L_{-1}|k; 0\rangle$. Show that:

(a) It can be written in the form

$$|\psi\rangle = \sqrt{2\alpha'} |k; k\rangle .$$

(b) It is orthogonal to any physical state.

(c) $L_1|\psi\rangle = 2\alpha' k^2 |k; 0\rangle .$

The $N = 1$ state $|k; \zeta\rangle$ constructed above with $k^2 = m^2 = 0$ and $k \cdot \zeta = 0$ has $d - 2 = 24$ independent polarization states, as the physical constraints remove two of the initial d vector degrees of freedom. It therefore describes a massless spin 1 (vector) particle with polarization vector ζ_μ , which agrees with what one finds for a massless Maxwell or Yang–Mills field. Indeed, there is a natural way to describe an associated gauge invariance in this picture.

Gauge Invariance : The corresponding spurious state $|\psi\rangle$ is both physical and null, and so we can add it to *any* physical state with no physical consequences. We should therefore impose an equivalence relation

$$|\text{phys}\rangle \sim |\text{phys}\rangle + \lambda |\psi\rangle \quad (3.20)$$

where λ is any constant. For the physical state $|\text{phys}\rangle = |k; \zeta\rangle$, Exercise 3.4 (a) implies that (3.20) is the same as an equivalence relation on

the polarization vectors,

$$\zeta^\mu \sim \zeta^\mu + \lambda \sqrt{2\alpha'} k^\mu . \quad (3.21)$$

This is a $U(1)$ gauge symmetry (in momentum space), and so at level $N = 1$ we have obtained the 24 physical states of a photon field $A_\mu(x)$ in 26-dimensional spacetime.

Higher Levels $N \geq 2$: The higher level string states with $N \geq 2$ are all massive and will not be dealt with here. We simply note that there is an infinite tower of them, thereby making the string theory suited to describe all of the elementary particles of nature.

3.2.2 The Closed String Spectrum

The case of closed strings is similar to that of open strings. We now have to also incorporate the left-moving sector Fock states. Thus we can easily extend the analysis to the closed string case by simply taking the tensor product of two copies of the open string result and changing the definition of the zero-modes as in (2.43). However, a new condition now arises. Adding and subtracting the two physical state conditions $(L_0 - 1)|\text{phys}\rangle = (\tilde{L}_0 - 1)|\text{phys}\rangle = 0$ yields, respectively, the quantum constraints

$$\begin{aligned} (L_0 + \tilde{L}_0 - 2)|\text{phys}\rangle &= 0 , \\ (L_0 - \tilde{L}_0)|\text{phys}\rangle &= 0 , \end{aligned} \quad (3.22)$$

where we have again fixed the value $a = 1$. The first constraint yields the usual mass-shell relation, since $H = L_0 + \tilde{L}_0 - 2$ is the worldsheet Hamiltonian which generates time translations on the string worldsheet. The second constraint can be understood by noting that the operator $P = L_0 - \tilde{L}_0$ is the worldsheet momentum, and so it generates translations in the string position coordinate σ . This constraint therefore simply reflects the fact that there is no physical significance as to where on the string we are, and hence that the physics is invariant under translations in σ . It amounts to equating the number of right-moving and left-moving oscillator modes. We thereby arrive at the new mass-shell relation

$$m^2 = \frac{4}{\alpha'} (N - 1) ,$$

(3.23)

and the additional “level-matching condition”

$$\boxed{N = \tilde{N}} . \quad (3.24)$$

Ground State $N = 0$: The ground state is $|k; 0, 0\rangle$ and it has mass-squared

$$m^2 = -\frac{4}{\alpha'} < 0 . \quad (3.25)$$

It again represents a spin 0 tachyon, and the closed string vacuum is also unstable.

First Excited Level $N = 1$: The first excited state is generically of the form

$$|k; \zeta\rangle = \zeta_{\mu\nu} \left(\alpha_{-1}^{\mu} |k; 0\rangle \otimes \tilde{\alpha}_{-1}^{\nu} |k; 0\rangle \right) \quad (3.26)$$

and it has mass-squared

$$m^2 = 0 . \quad (3.27)$$

The Virasoro constraints in addition give

$$L_1 |k; \zeta\rangle = \tilde{L}_1 |k; \zeta\rangle = 0 \quad (3.28)$$

which are equivalent to

$$k^{\mu} \zeta_{\mu\nu} = 0 . \quad (3.29)$$

A polarization tensor $\zeta_{\mu\nu}$ obeying (3.29) encodes three distinct spin states according to the decomposition of $\zeta_{\mu\nu}$ into irreducible representations of the spacetime “Little group” $SO(24)$, which classifies massless fields in this case [Figueroa-O’Farrill (2001)]. This is the residual Lorentz symmetry group that remains after the Virasoro constraints have been taken into account, so that the spectrum is exactly what one would get from 24 vectors in each sector which are transversely polarized to the light-cone. From a group theoretical perspective, this decomposition comes from writing down the Clebsch–Gordan decomposition of the tensor product of two vector representations **24** of $SO(24)$ into the irreducible symmetric, antisymmetric, and trivial representations,

$$\mathbf{24} \otimes \mathbf{24} = \mathbf{S} \oplus \mathbf{A} \oplus \mathbf{1} . \quad (3.30)$$

More concretely, we decompose the rank 2 tensor $\zeta_{\mu\nu}$ according to (3.30) as

$$\begin{aligned}\zeta_{\mu\nu} &= \left[\frac{1}{2} (\zeta_{\mu\nu} + \zeta_{\nu\mu}) - \frac{1}{25} \text{tr}(\zeta) \right] + \left[\frac{1}{2} (\zeta_{\mu\nu} - \zeta_{\nu\mu}) \right] \\ &\quad + \left[\frac{1}{25} \eta_{\mu\nu} \text{tr}(\zeta) \right] \\ &\equiv [g_{\mu\nu}] + [B_{\mu\nu}] + [\eta_{\mu\nu} \Phi] .\end{aligned}\quad (3.31)$$

The symmetric, traceless tensor $g_{\mu\nu}$ corresponds to the spin 2 “graviton field” and it yields the spacetime metric. The antisymmetric spin 2 tensor $B_{\mu\nu}$ is called the “Neveu–Schwarz B -field”, while the scalar field Φ is the spin 0 “dilaton”.

Gauge Invariance : The equivalence relations generated by spurious states built from L_{-1} and $\tilde{L}_{-1}$ give the “gauge transformations”

$$\begin{aligned}g_{\mu\nu} &\longmapsto g_{\mu\nu} + \partial_\mu \Lambda_\nu + \partial_\nu \Lambda_\mu , \\ B_{\mu\nu} &\longmapsto B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu .\end{aligned}\quad (3.32)$$

In this sense the tensor $g_{\mu\nu}$ admits a natural interpretation as a graviton field, as it has the correct diffeomorphism gauge invariance. Its presence accounts for the fact that string theory contains gravity, which is a good approximation at energies $\ll 1/\ell_s$. Its vacuum expectation value $\langle g_{\mu\nu} \rangle$ determines the spacetime geometry. Similarly, the vacuum expectation value of the dilaton Φ determines the “string coupling constant” g_s through

$$g_s = \langle e^\Phi \rangle . \quad (3.33)$$

This relationship can be derived using vertex operators, which will be discussed in the next section.

The gauge transformation rule for $B_{\mu\nu}$, on the other hand, is a generalization of that for the Maxwell field:

$$A_\mu \longmapsto A_\mu + \partial_\mu \Lambda . \quad (3.34)$$

The importance of the B -field resides in the fact that a fundamental string is a source for it, just like a charged particle is a source for an electromagnetic vector potential A_μ through the coupling

$$q \int d\tau \dot{x}^\mu(\tau) A_\mu . \quad (3.35)$$

In an analogous way, the B -field couples to strings via

$$q \int d^2\xi \epsilon^{ab} \partial_a x^\mu \partial_b x^\nu B_{\mu\nu} , \quad (3.36)$$

where ϵ^{ab} is the antisymmetric tensor with $\epsilon^{01} = -\epsilon^{10} = 1$.

3.2.3 *Worldsheet-Spacetime Interplay*

The upshot of the physical string spectrum can be summarized through the interplay between worldsheet and spacetime quantities. At the lowest level of massless states ($N = 1$), open strings correspond to gauge theory while closed strings correspond to gravity. This interplay will be a recurring theme in these notes. The higher levels $N \geq 2$ give an infinite tower of massive particle excitations. Notice that the massless states are picked out in the limit $\alpha' \rightarrow 0$ ($\ell_s \rightarrow 0$) in which the strings look like point-like objects. We shall refer to this low-energy limit as the “field theory limit” of the string theory.

3.3 Vertex Operators

Any local and unitary quantum field theory has an appropriate operator-state correspondence which allows one to associate quantum fields to quantum states in a one-to-one manner. The states may then be regarded as being created from the vacuum by the quantum fields. We will now describe how to formulate this correspondence in the context of string theory. For this, we map the closed string cylinder and the open string strip to the complex plane and the upper complex half-plane, respectively, first by a Wick rotation $\tau \mapsto i\tau$ to Euclidean worldsheet signature, followed by the coordinate transformation $z = e^{\tau - i\sigma}$ (Fig. 3.1). The advantage of this coordinate transformation is that it allows us to reinterpret the mode expansions of section 2.3 as Laurent series in the complex plane. In particular, for closed strings the coordinate transformation $\xi^\pm \mapsto z, \bar{z}$ allows us to write (2.38) as

$$\begin{aligned} \partial_z x_L^\mu(z) &= -i \sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \alpha_n^\mu z^{-n-1} , \\ \partial_{\bar{z}} x_R^\mu(\bar{z}) &= -i \sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \tilde{\alpha}_n^\mu \bar{z}^{-n-1} . \end{aligned} \quad (3.37)$$

These relations can be inverted by using the Cauchy integral formula to give

$$\begin{aligned}\alpha_{-n}^\mu &= \sqrt{\frac{2}{\alpha'}} \oint \frac{dz}{2\pi} z^{-n} \partial_z x_L^\mu(z) , \\ \tilde{\alpha}_{-n}^\mu &= \sqrt{\frac{2}{\alpha'}} \oint \frac{d\bar{z}}{2\pi} \bar{z}^{-n} \partial_{\bar{z}} x_R^\mu(\bar{z}) ,\end{aligned}\tag{3.38}$$

where the contour integrations encircle the origin $z = 0$ of the complex plane with counterclockwise orientation and are non-vanishing for $n \geq 0$. It follows that the states built from the α_{-n}^μ 's are related to the residues of $\partial_z^n x_L^\mu(z)$ at the origin, where $\partial_z^n x_L^\mu(0)$ corresponds to an insertion of a point-like operator at $z = 0$. These operators are called “vertex operators”. Let us consider some elementary examples of this operator-state correspondence.

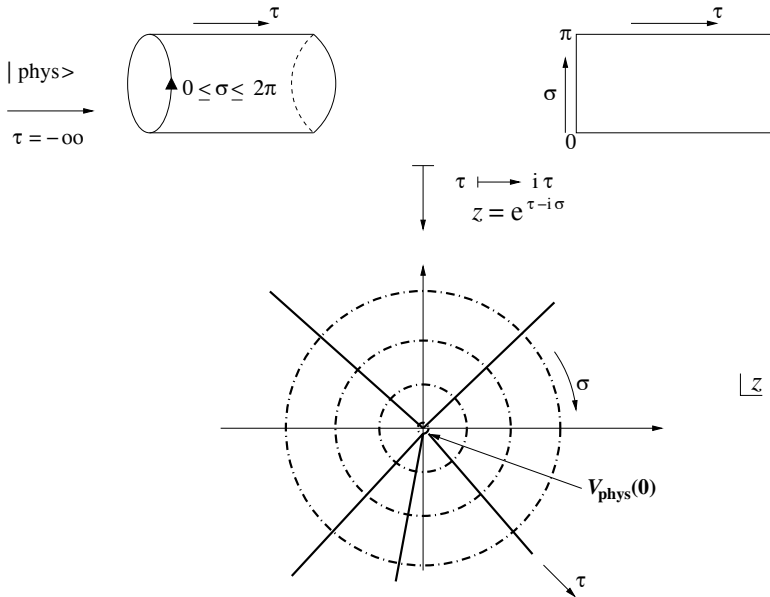


Fig. 3.1 The mapping of the worldsheet cylinder (resp. strip) onto the complex plane (resp. upper complex half-plane). The lines of constant τ form concentric circles about the origin in the complex z -plane, while the lines of constant σ correspond to rays emerging in the radial directions from the origin. An incoming physical state $|\text{phys}\rangle$ in the infinite worldsheet past ($\tau = -\infty$) corresponds to the insertion of a vertex operator $V_{\text{phys}}(z)$ at the origin $z = 0$.

3.3.1 Examples

Example 3.1. Geometrically, the closed string tachyon vertex operator is the spacetime Fourier transform of the operator

$$V(x) = \int d^2z \delta(x - x(z, \bar{z})) \quad (3.39)$$

which pins a string at the spacetime point x . In other words, the correspondence between tachyon ground states $|k; 0, 0\rangle = e^{ik \cdot x_0}$ and vertex operators is given by

$$|k; 0, 0\rangle \longleftrightarrow \int d^2z \, \circ \, e^{ik \cdot x(z, \bar{z})} \, \circ, \quad (3.40)$$

where $\circ \cdot \circ$ denotes normal ordering of quantum operators. This correspondence agrees with the anticipated behaviour under translations $x^\mu \mapsto x^\mu + c^\mu$ by constant vectors c^μ generated by the target space momentum, under which the operator (and state) pick up a phase factor $e^{ik \cdot c}$. Note that the state here is obtained by averaging over the absorption or emission point on the string worldsheet, as it should be independent of the particular insertion point.

Example 3.2. The emission or absorption of the gravitational fields $g_{\mu\nu}$, $B_{\mu\nu}$ and Φ are described via the operator-state correspondence

$$\zeta_{\mu\nu} \alpha_{-1}^\mu \tilde{\alpha}_{-1}^\nu |k; 0, 0\rangle \longleftrightarrow \int d^2z \, \zeta_{\mu\nu} \circ \partial_z x^\mu \partial_{\bar{z}} x^\nu e^{ik \cdot x} \circ, \quad (3.41)$$

defining the closed string level-1 vertex operator.

Example 3.3. The photon-emission vertex operator is the operator corresponding to the open string photon state given by the correspondence

$$\zeta_\mu \alpha_{-1}^\mu |k; 0\rangle \longleftrightarrow \int dl \, \zeta_\mu \circ \partial_{||} x^\mu e^{ik \cdot x} \circ, \quad (3.42)$$

where l is the coordinate of the real line representing the boundary of the upper half-plane (corresponding to the $\sigma = 0, \pi$ boundary lines of the worldsheet strip), and $\partial_{||}$ denotes the derivative in the direction tangential to the boundary. From this correspondence, which follows from the doubling trick $z \sim \bar{z}$ of section 2.3, it is evident that the photon is associated with the endpoints of the open string. This is a quantum version of the classical property that the string endpoints move at the speed of light, which can be easily deduced from examining the worldsheet canonical momentum of the open string theory.