

Lectures on Fuzzy and Fuzzy SUSY Physics

$$F *_C G(\vec{n}) = \sum_{m=0}^n \frac{(n-m)!}{m!n!} (\vec{e} \cdot \vec{n})^m \times K_{A_1 B_1}(\vec{n}) K_{A_2 B_2}(\vec{n}) \cdots$$

A. P. Balachandran | S. Kürkcüoğlu | S. Vaidya

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*Dedicated to Rafael Sorkin,
our friend, teacher,
a true and creative seeker of knowledge.*

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Preface

One of us (Balachandran) gave a course of lectures on “Fuzzy Physics” during spring, 2002 for students of Syracuse and Brown Universities. The course which used video conferencing technology was also put on the websites [1]. Subsequently A.P. Balachandran, S. K rk o lu and S.Vaidya decided to edit the material and publish them as lecture notes. The present book is the outcome of that effort.

The recent interest in fuzzy physics begins from the work of Madore [2, 3] even though the basic physical and mathematical ideas are older and go back to Hoppe [4]. Many of the mathematical developments are based on the works of Kostant and Kirillov [5] and Berezin [6]. They emerge from the fundamental observation that coadjoint orbits of Lie groups are symplectic manifolds which can therefore be quantized under favorable circumstances. When that can be done, we get a quantum representation of the manifold. It is the fuzzy manifold for the underlying “classical manifold”. It is fuzzy because no precise localization of points thereon is possible. The fuzzy manifold approaches its classical version when the effective Planck’s constant of quantization goes to zero.

Our interest will be in compact simple and semi-simple Lie groups for which coadjoint and adjoint orbits can be identified and are compact as well. In such a case these fuzzy manifold is a finite-dimensional matrix algebra on which the Lie group acts in simple ways. Such fuzzy spaces are therefore very simple and also retain the symmetries of their classical spaces. These are some of the reasons for their attraction.

There are several reasons to study fuzzy manifolds. Our interest has its roots in quantum field theory (qft). Qft’s require regularization and the conventional nonperturbative regularization is lattice regularization. It has been extensively studied for over thirty years. It fails to preserve

space-time symmetries of quantum fields. It also has problems in dealing with topological subtleties like instantons, and can deal with index theory and axial anomaly only approximately. Instead fuzzy physics does not have these problems. So it merits investigation as an alternative tool to investigate qft's.

A related positive feature of fuzzy physics, is its ability to deal with supersymmetry(SUSY) in a precise manner [7–10]. (See however,[11]). Fuzzy SUSY models are also finite-dimensional matrix models amenable to numerical work, so this is another reason for our attraction to this field.

Interest in fuzzy physics need not just be utilitarian. Physicists have long speculated that space-time in the small has a discrete structure. Fuzzy space-time gives a very concrete and interesting method to model this speculation and test its consequences. There are many generic consequences of discrete space-time, like CPT and causality violations, and distortions of the Planck spectrum. Among these must be characteristic signals for fuzzy physics, but they remain to be identified.

We are a part of a collaboration on fuzzy physics and noncommutative geometry. These lecture notes have been strongly influenced by our interactions with our colleagues in this collaboration. We thank them for discussions and ideas. The work of APB has been supported in part by DOE under contract number DE-FG02-85ER40231. The work of SK is supported by Irish Research Council for Science Engineering and Technology(IRCSET).

These lecture notes are not exhaustive, and reflect the research interests of the authors. It is our hope that the interested reader will be able to learn about the topics we have not covered with the help of our citations.

Contents

<i>Preface</i>	vii
1. Introduction	1
2. Fuzzy Spaces	5
2.1 Fuzzy $\mathbb{C}^2$	5
2.2 Fuzzy S^3 and Fuzzy S^2	5
2.3 The Fuzzy Sphere S_F^2	6
2.4 Observables of S_F^2	8
2.5 Diagonalizing $\mathcal{L}^2$	9
2.6 Scalar Fields on S_F^2	9
2.7 The Holstein-Primakoff Construction	10
2.8 $\mathbb{C}P^N$ and Fuzzy $\mathbb{C}P^N$	11
2.9 The $\mathbb{C}P^N$ Holstein-Primakoff Construction	14
3. Star Products	17
3.1 Introduction	17
3.2 Properties of Coherent States	19
3.3 The Coherent State or Voros $\ast$ -Product on the Moyal Plane	20
3.4 The Moyal $\ast$ -Product on the Groenewold-Moyal Plane	23
3.4.1 The Weyl Map and the Weyl Symbol	23
3.5 Properties of $\ast$ -Products	24
3.5.1 Cyclic Invariance	24
3.5.2 A Special Identity for the Weyl Star	25
3.5.3 Equivalence of $\ast_C$ and $\ast_W$	25

3.5.4	Integration and Tracial States	26
3.5.5	The θ -Expansion	27
3.6	The $*$ -Products for the Fuzzy Sphere	28
3.6.1	The Coherent State $*$ -Product $*_C$	28
3.6.2	The Weyl $*$ -Product $*_W$	31
4.	Scalar Fields on the Fuzzy Sphere	35
4.1	Loop Expansion	37
4.2	The One-Loop Two-Point Function	39
4.3	Numerical Simulations	43
4.3.1	Scalar Field Theory on Fuzzy S^2	45
4.3.2	Gauge Theory on Fuzzy $S^2 \times S^2$	46
5.	Instantons, Monopoles and Projective Modules	47
5.1	Free Modules, Projective Modules	47
5.2	Projective Modules on $\mathcal{A} = C^\infty(S^2)$	49
5.3	Equivalence of Projective Modules	51
5.4	Projective Modules on the Fuzzy Sphere	54
5.4.1	Fuzzy Monopoles and Projectors $P_F^{(k)}$	54
5.4.2	The Fuzzy Module for the Tangent Bundle and the Fuzzy Complex Structure	56
6.	Fuzzy Nonlinear Sigma Models	59
6.1	Introduction	59
6.2	CP^1 Models and Projectors	60
6.3	An Action	63
6.4	$\mathbb{C}P^1$ -Models and Partial Isometries	65
6.4.1	Relation Between $\mathcal{P}^{(\kappa)}$ and $\mathcal{P}_\kappa$	68
6.5	Fuzzy $\mathbb{C}P^1$ -Models	69
6.5.1	The Fuzzy Projectors for $\kappa > 0$	69
6.5.2	The Fuzzy Projectors for $\kappa < 0$	70
6.5.3	The Fuzzy Winding Number	71
6.5.4	The Generalized Fuzzy Projector : Duality or BPS States	71
6.5.5	The Fuzzy Bound	71
6.6	$\mathbb{C}P^N$ -Models	73

7.	Fuzzy Gauge Theories	77
7.1	Limits on Gauge Groups	78
7.2	Limits on Representations of Gauge Groups	79
7.3	Connection and Curvature	80
7.4	Instanton Sectors	81
7.5	The Partition Function and the θ -parameter	82
8.	The Dirac Operator and Axial Anomaly	85
8.1	Introduction	85
8.2	A Review of the Ginsparg-Wilson Algebra	85
8.3	Fuzzy Models	88
8.3.1	Review of the Basic Fuzzy Sphere Algebra	88
8.3.2	The Fuzzy Dirac Operator (No Instantons or Gauge Fields)	89
8.3.3	The Fuzzy Gauged Dirac Operator (No Instanton Fields)	91
8.4	The Basic Instanton Coupling	93
8.4.1	Mixing of Spin and Isospin	94
8.4.2	The Spectrum of the Dirac operator	94
8.5	Gauging the Dirac Operator in Instanton Sectors	95
8.6	Further Remarks on the Axial Anomaly	96
9.	Fuzzy Supersymmetry	99
9.1	$osp(2,1)$ and $osp(2,2)$ Superalgebras and their Representations	100
9.2	Passage to Supergroups	106
9.3	On the Superspaces	107
9.3.1	The Superspace $\mathcal{C}^{2,1}$ and the Noncommutative $\mathcal{C}_F^{2,1}$	107
9.3.2	The Supersphere $S^{(3,2)}$ and the Noncommutative $S^{(3,2)}$	108
9.3.3	The Commutative Supersphere $S^{(2,2)}$	108
9.3.4	The Fuzzy Supersphere $S_F^{(2,2)}$	112
9.4	More on Coherent States	114
9.5	The Action on the Supersphere $S^{(2,2)}$	116
9.6	The Action on the Fuzzy Supersphere $S_F^{(2,2)}$	119
9.6.1	The Integral and Supertrace	119
9.6.2	$OSp(2,1)$ IRR's with Cut-Off N	121

9.6.3	The Highest Weight States and the $osp(2,2)$ -Invariant Action	121
9.6.4	The Spectrum of V	122
9.6.5	The Fuzzy SUSY Action	124
9.7	The $*$ -Products	125
9.7.1	The $*$ -Product on $S_F^{(2,2)}$	125
9.7.2	The $*$ -Product on Fuzzy “Sections of Bundles”	127
9.8	More on the Properties of $\mathcal{K}_{ab}$	129
9.9	The $O(3)$ Nonlinear Sigma Model on $S^{(2,2)}$	131
9.9.1	The Model on $S^{(2,2)}$	131
9.9.2	The Model on $S_F^{(2,2)}$	132
9.9.3	Supersymmetric Extensions of Bott Projectors	133
9.9.4	The SUSY Action Revisited	134
9.9.5	Fuzzy Projectors and Sigma Models	135
10.	SUSY Anomalies on the Fuzzy Supersphere	137
10.1	Overview	137
10.1.1	The Fuzzy Sphere	137
10.1.2	SUSY	138
10.1.3	Irreducible Representations	139
10.1.4	Casimir Operators	140
10.1.5	Tensor Products	141
10.1.6	The Supertrace and the Grade Adjoint	141
10.1.7	The Free Action	141
10.2	SUSY Chirality	143
10.3	Eigenvalues of V_0	144
10.4	Fuzzy SUSY Instantons	145
10.5	Fuzzy SUSY Zero Modes and their Index Theory	146
10.5.1	Spectrum of K_2	147
10.5.2	Index Theory and Zero Modes	149
10.6	Final Remarks	149
11.	Fuzzy Spaces as Hopf Algebras	151
11.1	Overview	151
11.2	Basics	153
11.3	The Group and the Convolution Algebras	154
11.4	A Prelude to Hopf Algebras	155

11.5 The $\ast$ -Homomorphism $G^\ast \rightarrow S_F^2$ 159

11.6 Hopf Algebra for the Fuzzy Spaces 161

11.7 Interpretation 165

11.8 The Prešnajder Map 166

Bibliography 169

Index 179

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Chapter 1

Introduction

We can find few fundamental physical models amenable to exact treatment. Approximation methods like perturbation theory are necessary and are part of our physics culture.

Among the important approximation methods for quantum field theories (qft's) are strong coupling methods based on lattice discretization of the underlying spacetime or perhaps its time-slice. They are among the rare effective approaches for the study of confinement in QCD and for non-perturbative regularization of qft's. They enjoyed much popularity in their early days and have retained their good reputation for addressing certain fundamental problems.

One feature of naive lattice discretizations however can be criticized. They do not retain the symmetries of the exact theory except in some rough sense. A related feature is that topology and differential geometry of the underlying manifolds are treated only indirectly, by limiting the couplings to “nearest neighbors”. Thus lattice points are generally manipulated like a trivial topological set, with a point being both open and closed. The upshot is that these models have no rigorous representation of topological defects and lumps like vortices, solitons and monopoles. The complexities in the ingenious lattice representations of the QCD θ -term [12] illustrate such limitations. There do exist radical attempts to overcome these limitations using partially ordered sets [13], but their potentials are yet to be adequately studied.

As mentioned in the preface, a new approach to discretization, under the name of “fuzzy physics” inspired by noncommutative geometry (NCG), is being developed for a while now. The key remark here is that when the underlying spacetime or spatial cut can be treated as a phase space and quantized, with a parameter $\hat{h}$ assuming the role of $\hbar$, the emergent quan-

tum space is fuzzy, and the number of independent states per (“classical”) unit volume becomes finite. We have known this result after Planck and Bose introduced such an ultraviolet cut-off and quantum physics later justified it. A “fuzzified” manifold is expected to be ultraviolet finite, and if the parent manifold is compact too, supports only finitely many independent states. The continuum limit is the semi-classical $\hbar \rightarrow 0$ limit. This unconventional discretization of classical topology is not at all equivalent to the naive one, and we shall see that it does significantly overcome the previous criticisms.

There are other reasons also to pay attention to fuzzy spaces, be they spacetimes or spatial slices. There is much interest among string theorists in matrix models and in describing D -branes using matrices. Fuzzy spaces lead to matrix models too and their ability to reflect topology better than elsewhere should therefore evoke our curiosity. They let us devise new sorts of discrete models and are interesting from that perspective. In addition, as mentioned in the preface, it has now been discovered that when open strings end on D -branes which are symplectic manifolds, then the branes can become fuzzy. In this way one comes across fuzzy tori, $\mathbb{C}P^N$ and many such spaces in string physics.

The central idea behind fuzzy spaces is discretization by quantization. It does not always work. An obvious limitation is that the parent manifold has to be even dimensional. If it is not, it has no chance of being a phase space. But that is not all. Successful use of fuzzy spaces for qft’s requires good fuzzy versions of the Laplacian, Dirac equation, chirality operator and so forth, and their incorporation can make the entire enterprise complicated. The torus T^2 is compact, admits a symplectic structure and on quantization becomes a fuzzy, or a non-commutative torus. It supports a finite number of states if the symplectic form satisfies the Dirac quantization condition. But it is impossible to introduce suitable derivations without escalating the formalism to infinite dimensions.

But we do find a family of classical manifolds elegantly escaping these limitations. They are the co-adjoint orbits of Lie groups. For semi-simple Lie groups, they are the same as adjoint orbits. It is a theorem that these orbits are symplectic. They can often be quantized when the symplectic forms satisfy the Dirac quantization condition. The resultant fuzzy spaces are described by linear operators on irreducible representations (IRR’s) of the group. For compact orbits, the latter are finite-dimensional. In addition, the elements of the Lie algebra define natural derivations, and that helps to find the Laplacian and the Dirac operator. We can even

define chirality with no fermion doubling and represent monopoles and instantons. (See chapters 5, 6 and 8). These orbits therefore are altogether well-adapted for qft's.

Let us give examples of these orbits:

- $S^2 \simeq \mathbb{CP}^1$: This is the orbit of $SU(2)$ through the Pauli matrix σ_3 or any of its multiples $\lambda \sigma_3$ ($\lambda \neq 0$). It is the set $\{\lambda g \sigma_3 g^{-1} : g \in SU(2)\}$. The symplectic form is $j d(\cos) \theta \wedge d\phi$ with θ, ϕ being the usual S^2 coordinates. Quantization gives the spin j $SU(2)$ representations.
- $\mathbb{CP}^2$: $\mathbb{CP}^2$ is of particular interest being of dimension 4. It is the orbit of $SU(3)$ through the hypercharge $Y = 1/3 \text{ diag}(1, 1, -2)$ (or its non-zero multiples):

$$\mathbb{CP}^2 : \{g Y g^{-1} : g \in SU(3)\}. \quad (1.1)$$

The associated representations are symmetric products of 3 's or $\bar{3}$'s.

In a similar way $\mathbb{CP}^N$ are adjoint orbits of $SU(N+1)$ for any $N \geq 3$. They too can be quantized and give rise to fuzzy spaces.

- $SU(3)/[U(1) \times U(1)]$: This 6-dimensional manifold is the orbit of $SU(3)$ through $\lambda_3 = \text{diag}(1, -1, 0)$ and its non-zero multiples. These orbits give all the IRR's containing a zero hypercharge state.

In this book, we focus on the fuzzy spaces emerging from quantizing S^2 . They are called the fuzzy spheres S_F^2 and depend on the integer or half integer j labelling the irreducible representations of $SU(2)$. Physics on S_F^2 is treated in detail. Scalar and gauge fields, the Dirac operator, instantons, index theory, and the so-called UV-IR mixing [14–20] are all covered. Supersymmetry can be elegantly discretized in the approach of fuzzy physics by replacing the Lie algebra $su(2)$ of $SU(2)$ by the superalgebras $osp(2, 1)$ and $osp(2, 2)$. Fuzzy supersymmetry is also discussed in these lectures including its instanton and index theories. We also briefly discuss the fuzzy spaces associated with $\mathbb{CP}^N$ ($N \geq 2$). These spaces, especially $\mathbb{CP}^2$, are of physical interest. We refer to the literature [29–33] for their more exhaustive treatment.

Fuzzy physics draws from many techniques and notions developed in the context of noncommutative geometry. There are excellent books and reviews on this vast subject some of which we include in the bibliography [3, 34–38].

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Chapter 2

Fuzzy Spaces

In the present chapter, we approach the problem of quantization of classical manifolds like S^2 and $\mathbb{CP}^N$ using harmonic oscillators. The method is simple and transparent, and enjoys generality too. The point of departure in this approach is the quantization of complex planes. We focus on quantizing $\mathbb{C}^2$ and its associated S^2 first. We will consider other manifolds later in the chapter.

2.1 Fuzzy $\mathbb{C}^2$

The two-dimensional complex plane $\mathbb{C}^2$ has coordinates $z = (z_1, z_2)$ where $z_i \in \mathbb{C}$. We want to quantize $\mathbb{C}^2$ turning it into fuzzy $\mathbb{C}^2 \equiv \mathbb{C}_F^2$.

This is easily accomplished. After quantization, z_i become harmonic oscillator annihilation operators a_i and z_i^* become their adjoint. Their commutation relations are

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0, \quad [a_i, a_j^\dagger] = \tilde{\hbar} \delta_{ij}, \quad (2.1)$$

where the $\tilde{\hbar}$ need not be the “Planck’s constant/ 2π ”. The classical manifold emerges as $\tilde{\hbar} \rightarrow 0$. We set the usual Planck’s constant $\hbar$ to 1 hereafter unless otherwise stated.

In the same way, we can quantize $\mathbb{C}^{N+1}$ for any N using an appropriate number of oscillators and that gives us fuzzy $\mathbb{CP}^N$ as we shall later see.

2.2 Fuzzy S^3 and Fuzzy S^2

There is a well-known descent chain from $\mathbb{C}^2$ to the 3-sphere S^3 and thence to S^2 . Our tactic to obtain fuzzy $S^2 \equiv S_F^2$ is to quantize this chain,

obtaining along the way fuzzy $S^3 \equiv S_F^3$.

Let us recall this chain of manifolds. Consider $\mathbb{C}^2$ with the origin removed, $\mathbb{C}^2 \setminus \{0\}$. As $z \neq 0$, $\frac{z}{|z|}$ with $|z| = (\sum_i |z_i|^2)^{\frac{1}{2}}$ makes sense here. Since $\frac{z}{|z|}$ is normalized to 1, $|\frac{z}{|z|}| = 1$, it gives the 3-sphere S^3 . Thus we have the fibration

$$\mathbb{R} \rightarrow \mathbb{C}^2 \setminus \{0\} \rightarrow S^3 = \left\langle \xi = \frac{z}{|z|} \right\rangle, \quad z \rightarrow \frac{z}{|z|}. \quad (2.2)$$

Now S^3 is a $U(1)$ -bundle ("Hopf fibration") [45] over S^2 . If $\xi \in S^3$, then $\vec{x}(\xi) = \xi^\dagger \vec{\tau} \xi$ (where $\tau_i, i = 1, 2, 3$ are the Pauli matrices) is invariant under the $U(1)$ action $\xi \rightarrow \xi e^{i\theta}$ and is a real normalized three-vector:

$$\vec{x}(\xi)^* = \vec{x}(\xi), \quad \vec{x}(\xi) \cdot \vec{x}(\xi) = 1. \quad (2.3)$$

So $\vec{x}(\xi) \in S^2$ and we have the Hopf fibration

$$U(1) \rightarrow S^3 \rightarrow S^2, \quad \xi \rightarrow \vec{x}(\xi). \quad (2.4)$$

Note that $\vec{x}(\xi) = \frac{1}{|z|} z^* \vec{\tau} z \frac{1}{|z|}$.

The fuzzy S^3 is obtained by replacing $\frac{z_i}{|z|}$ by $a_i \frac{1}{\sqrt{\hat{N}}}$ where $\hat{N} = \sum_j a_j^\dagger a_j$ is the number operator:

$$\frac{z_i}{|z|} \rightarrow a_i \frac{1}{\sqrt{\hat{N}}}, \quad \frac{z_i^*}{|z|} \rightarrow \frac{1}{\sqrt{\hat{N}}} a_i^\dagger, \quad \hat{N} = \sum_j a_j^\dagger a_j, \quad \hat{N} \neq 0. \quad (2.5)$$

The quantum condition $\hat{N} \neq 0$ means that the vacuum is omitted from the Hilbert space, so that it is the orthogonal complement of the vacuum in Fock space. This omission is like the deletion of 0 from $\mathbb{C}^2$.

There is a problem with this omission as $a_i \frac{1}{\sqrt{\hat{N}}}$ and its polynomials will create it from any $\hat{N} = n$ state. For this reason, and because $a_i \frac{1}{\sqrt{\hat{N}}}$ and its adjoint need the infinite-dimensional Fock space to act on and do not give finite-dimensional models for S_F^3 , we will not dwell on this space.

2.3 The Fuzzy Sphere S_F^2

The problems of S_F^3 melt away for S_F^2 . Quantization of S^2 gives S_F^2 with $x_i(\xi)$ becoming the operator x_i :

$$x_i(\xi) \rightarrow x_i = \frac{1}{\sqrt{\hat{N}}} a^\dagger \vec{\tau} a \frac{1}{\sqrt{\hat{N}}} = \frac{1}{\hat{N}} a^\dagger \tau_i a, \quad \hat{N} \neq 0. \quad (2.6)$$

Since

$$[x_i, \hat{N}] = 0, \quad (2.7)$$

we can restrict x_i to the subspace $\mathcal{H}_n$ of the Fock space where $\hat{N} = n$ ($\neq 0$). This space is $(n+1)$ -dimensional and is spanned by the orthonormal vectors

$$\frac{(a_1^\dagger)^{n_1}}{\sqrt{n_1!}} \frac{(a_2^\dagger)^{n_2}}{\sqrt{n_2!}} |0\rangle \equiv |n_1 n_2\rangle, \quad n_1 + n_2 = n. \quad (2.8)$$

The x_i act irreducibly on this space and generate the full matrix algebra $Mat(n+1)$.

The $SU(2)$ angular momentum operators L_i are given by the Schwinger construction:

$$L_i = a^\dagger \frac{\tau_i}{2} a, \quad [L_i, L_j] = i\epsilon_{ijk} L_k. \quad (2.9)$$

$a_i^\dagger$ transform as spin $\frac{1}{2}$ spinors and (2.8) spans the n -fold symmetric product of these spinors. It has angular momentum $\frac{n}{2}$:

$$L_i L_i |_{\mathcal{H}_n} = \frac{n}{2} \left(\frac{n}{2} + 1 \right) \mathbf{1} |_{\mathcal{H}_n}. \quad (2.10)$$

Since

$$x_i |_{\mathcal{H}_n} = \frac{2}{n} L_i |_{\mathcal{H}_n}, \quad (2.11)$$

we find

$$\begin{aligned} [x_i, x_j] |_{\mathcal{H}_n} &= \frac{2}{n} i\epsilon_{ijk} x_k |_{\mathcal{H}_n}, \\ \left(\sum_i x_i^2 \right) |_{\mathcal{H}_n} &= \left(1 + \frac{2}{n} \right) \mathbf{1} |_{\mathcal{H}_n}. \end{aligned} \quad (2.12)$$

S_F^2 has radius $(1 + \frac{2}{n})^{\frac{1}{2}}$ which becomes 1 as $n \rightarrow \infty$.

We generally write the equations in (2.12) as $[x_i, x_j] = \frac{2}{n} i\epsilon_{ijk} x_k$, $(\sum_i x_i^2) = (1 + \frac{2}{n})$, omitting the indication of $\mathcal{H}_n$. S_F^2 should have an additional label n , but that too is usually omitted. The x_i 's are seen to commute in the naive continuum limit $n \rightarrow \infty$ giving back the commutative algebra of functions on S^2 .

The fuzzy sphere S_F^2 is a “quantum” object. It has wave functions which are generated by x_i restricted to $\mathcal{H}_n$. Its Hilbert space is $Mat(n+1)$ with the scalar product

$$(m_1, m_2) = \frac{1}{n+1} Tr m_1^\dagger m_2, \quad m_i \in Mat(n+1). \quad (2.13)$$

We denote $Mat(n+1)$ with this scalar product also as $Mat(n+1)$.

2.4 Observables of S_F^2

The observables of S_F^2 are associated with linear operators on $Mat(n+1)$. We can associate two linear operators α^L and α^R to each $\alpha \in Mat(n+1)$. They have left- and right-actions on $Mat(n+1)$;

$$\alpha^L m = \alpha m, \quad \alpha^R m = m \alpha, \quad \forall \quad m \in Mat(n+1) \quad (2.14)$$

and fulfill

$$(\alpha\beta)^L = \alpha^L \beta^L, \quad (\alpha\beta)^R = \beta^R \alpha^R. \quad (2.15)$$

Such left- and right- operators commute:

$$[\alpha^L, \beta^R] = 0, \quad \forall \quad \alpha, \beta \in Mat(n+1). \quad (2.16)$$

We denote the two commuting matrix algebras of left- and right- operators by $Mat_{L,R}(n+1)$. $Mat(n+1)$ is generated by $a_i^\dagger a_j$ with the understanding that their domain is $\mathcal{H}_n$. Accordingly, $Mat_{L,R}(n+1)$ are generated by $(a_i^\dagger a_j)^{L,R}$.

We can also define operators $a_i^{L,R}, (a_j^\dagger)^{L,R}$:

$$\begin{aligned} a_i^L m &= a_i m, & a_i^R m &= m a_i \\ a_j^{\dagger L} m &= a_j^\dagger m, & a_j^{\dagger R} m &= m a_j^\dagger. \end{aligned} \quad (2.17)$$

They are operators changing n :

$$\begin{aligned} a_i^{L,R} &: \quad \mathcal{H}_n \rightarrow \mathcal{H}_{n-1} \\ a_j^{\dagger L,R} &: \quad \mathcal{H}_n \rightarrow \mathcal{H}_{n+1}. \end{aligned} \quad (2.18)$$

Such operators are important for discussions of bundles. (See chapter 5.)

With the help of these operators, we can write

$$(a_i^\dagger a_j)^L = a_i^{\dagger L} a_j^L, \quad (a_i^\dagger a_j)^R = a_j^R a_i^{\dagger R}. \quad (2.19)$$

Of particular interest are the three angular momentum operators

$$L_i^L, L_i^R, \quad \mathcal{L}_i = L_i^L - L_i^R. \quad (2.20)$$

Of these, $\mathcal{L}_i$ annihilates $\mathbf{1}$ as does the continuum orbital angular momentum. It is the fuzzy sphere angular momentum approaching the orbital angular momentum of S^2 as $n \rightarrow \infty$:

$$\mathcal{L}_i \rightarrow -i(\vec{x}(\xi) \wedge \vec{\nabla}) \equiv -i\epsilon_{ijk} x(\xi)_j \frac{\partial}{\partial x(\xi)_k} \quad \text{as } n \rightarrow \infty. \quad (2.21)$$

2.5 Diagonalizing $\mathcal{L}^2$

We have $\sum_i (L_i^L)^2 = \sum_i (L_i^R)^2 = \frac{n}{2}(\frac{n}{2} + 1)$ so that orbital angular momentum is the sum of two angular momenta with values $\frac{n}{2}$. Hence the spectrum of $\mathcal{L}^2$ is

$$\langle \ell(\ell + 1) : \ell \in \{0, 1, 2, \dots, n\} \rangle. \quad (2.22)$$

A function f in $C^\infty(S^2)$ has the expansion

$$f = \sum_{\ell, m} a_{\ell m} Y_{\ell m} \quad (2.23)$$

in terms of the spherical harmonics $Y_{\ell m}$. The spectrum of orbital angular momentum is thus $\langle \ell(\ell + 1) : \ell \in \{0, 1, 2, \dots, n, \dots\} \rangle$.

The spectrum of $\mathcal{L}^2$ is thus precisely that of the continuum orbital angular momentum cut off at n . There is no distortion of eigenvalues upto n .

The eigenstates $T_m^\ell, m \in \{-\ell, -\ell + 1, \dots, \ell\}$ of $\mathcal{L}^2$ are known as polarization tensors [46]. They are eigenstates of $\mathcal{L}_3$ and also orthonormal:

$$\begin{aligned} \mathcal{L}^2 T_m^\ell &= \ell(\ell + 1) T_m^\ell, \\ \mathcal{L}_3 T_m^\ell &= m T_m^\ell, \\ (T_{m'}^{\ell'}, T_m^\ell) &= \delta_{\ell\ell'} \delta_{m'm}. \end{aligned} \quad (2.24)$$

2.6 Scalar Fields on S_F^2

We will be brief here as they are treated in detail in chapter 4. A complex scalar field Φ on S^2 is a power series in the coordinate functions $m_i := x_i$,

$$\Phi = \sum a_{i_1 \dots i_n} m_{i_1} \cdots m_{i_n}. \quad (2.25)$$

(Note again that $\vec{m} \cdot \vec{m} = 1$) The Laplacian on S^2 is $\Delta := -(-i\vec{x} \wedge \vec{\nabla})^2$ and a simple Euclidean action is

$$\mathcal{S} = - \int \frac{d\Omega}{4\pi} \Phi^* \Delta \Phi \quad d\Omega = d \cos(\theta) d\phi. \quad (2.26)$$

We can simplify (2.26) by the expansion

$$\Phi(\vec{x}) = \sum \Phi_{\ell m} Y_{\ell m}(\vec{m}). \quad (2.27)$$

Then since $\Delta Y_{\ell m}(\vec{m}) = -\ell(\ell + 1) Y_{\ell m}(\vec{m})$, and $\int \frac{d\Omega}{4\pi} Y_{\ell' m'}(\vec{m})^* Y_{\ell m}(\vec{m}) = \delta_{\ell\ell'} \delta_{mm'}$,

$$\mathcal{S} = \sum_{\ell, m} \ell(\ell + 1) \Phi_{\ell m}^* \Phi_{\ell m}. \quad (2.28)$$

From (2.25), we infer that the fuzzy scalar field Ψ is a power series in the matrices x_i and is hence itself a matrix. The Euclidean action replacing (2.26) is

$$S = (\mathcal{L}_i \Psi, \mathcal{L}_i \Psi) = -(\Psi, \Delta \Psi). \quad (2.29)$$

On expanding ψ according to

$$\Psi = \sum_{\ell \leq n+1} \psi_{\ell m} T_m^\ell, \quad (2.30)$$

this reduces to

$$S = \sum_{\ell \leq n} \ell(\ell+1) |\psi_{\ell m}|^2. \quad (2.31)$$

2.7 The Holstein-Primakoff Construction

There is an interesting construction of L_i for fixed n using just one oscillator due to Holstein and Primakoff [47]. We outline this construction here.

In brief, since $\hat{N}$ commutes with L_i , we can eliminate a_2 from L_i and restrict L_i to $\mathcal{H}_n$ without spoiling their commutation relations. The result is the Holstein-Primakoff construction.

We now give the details. (2.5) gives the following polar decomposition of a_2 :

$$a_2 = U \sqrt{N - a_1^\dagger a_1}, \quad U^\dagger U = U U^\dagger = \mathbf{1}, \quad (2.32)$$

where we choose the positive square root:

$$\sqrt{N - a_1^\dagger a_1} \geq 0. \quad (2.33)$$

We can understand U better by examining the action of a_2 on the orthonormal states (2.8) spanning $\mathcal{H}_n$. We find

$$\begin{aligned} a_2 |n_1, n_2\rangle &= \sqrt{n_2} |n_1, n_2 - 1\rangle \\ &= U \sqrt{N - a_1^\dagger a_1} |n_1, n_2\rangle \\ &= \sqrt{n_2} U |n_1, n_2\rangle \end{aligned} \quad (2.34)$$

or

$$U |n_1, n_2\rangle = |n_1, n_2 - 1\rangle \quad (2.35)$$

Thus if

$$A^\dagger = a_1^\dagger U, \quad A = U^\dagger a_1, \quad (2.36)$$

then

$$\begin{aligned} A^\dagger |n_1, n_2\rangle &= \sqrt{n_1 + 1} |n_1 + 1, n_2 - 1\rangle, \\ A |n_1, n_2\rangle &= \sqrt{n_1} |n_1 - 1, n_2 - 1\rangle, \end{aligned} \quad (2.37)$$

and

$$\begin{aligned} [A, A^\dagger] &= \mathbf{1}, \quad [A^\dagger, A^\dagger] = [A, A] = 0, \\ [A, N] &= [A^\dagger, N] = 0. \end{aligned} \quad (2.38)$$

a_2 vanishes on $|n, 0\rangle$ and U and $A^\dagger$ are undefined on that vector. That is, A and $A^\dagger$ can not be defined on $\mathcal{H}_n$. In any case, the oscillator algebra of (2.38) has no finite-dimensional representation. But this is not the case for L_i . We have

$$\begin{aligned} L_+ &= L_1 + iL_2 = a_1^\dagger a_2 = A^\dagger \sqrt{N - A^\dagger A} \\ L_- &= L_1 - iL_2 = a_2^\dagger a_1 = \sqrt{N - A^\dagger A} A \\ L_3 &= a_1^\dagger a_1 - a_2^\dagger a_2 = A^\dagger A - N \end{aligned} \quad (2.39)$$

On $\mathcal{H}_n$, (2.39) gives the Holstein-Primakoff realization of the $SU(2)$ Lie algebra for angular momentum $\frac{n}{2}$.

2.8 $\mathbb{C}P^N$ and Fuzzy $\mathbb{C}P^N$

S^2 is $\mathbb{C}P^1$ as a complex manifold. The additional structure for $\mathbb{C}P^1$ as compared to S^2 is only the complex structure. So we can without great harm denote S^2 and S_F^2 also as $\mathbb{C}P^1$ and $\mathbb{C}P_F^1$. In section 5.4.2, we will in fact consider the complex structure and its quantization.

Generalizations of $\mathbb{C}P^1$ and $\mathbb{C}P_F^1$ are $\mathbb{C}P^N$ and $\mathbb{C}P_F^N$. They are associated with the groups $SU(N+1)$.

Classically $\mathbb{C}P^N$ is the complex projective space of complex dimension N . It can be described as follows. Consider the $(2N+1)$ -dimensional sphere

$$S^{2N+1} = \left\langle \xi = (\xi_1, \xi_2, \dots, \xi_{N+1}) : \xi_i \in \mathbb{C}, |\xi|^2 := \sum |\xi_i|^2 = 1 \right\rangle. \quad (2.40)$$

It admits the $U(1)$ action

$$\xi \rightarrow e^{i\theta} \xi. \quad (2.41)$$

$\mathbb{C}P^N$ is the quotient of S^{2N+1} by this action giving rise to the fibration

$$U(1) \rightarrow S^{2N+1} \rightarrow \mathbb{C}P^N. \quad (2.42)$$

If λ_i are the Gell-Mann matrices of $SU(N+1)$, a point of $\mathbb{C}P^N$ is

$$\vec{X}(\xi) = \xi^\dagger \vec{\lambda} \xi, \quad \xi \in S^{2N+1}. \quad (2.43)$$

For $N = 1$, these become the previously constructed structures.

There is another description of S^{2N+1} and $\mathbb{CP}^N$. $SU(N+1)$ acts transitively on S^{2N+1} and the stability group at $(1, \vec{0})$ is

$$SU(N) = \left\langle u \in SU(N+1) : u = \begin{pmatrix} 1 & 0 \\ 0 & \hat{u} \end{pmatrix} \right\rangle. \quad (2.44)$$

Hence

$$S^{2N+1} = SU(N+1)/SU(N). \quad (2.45)$$

Consider the equivalence class

$$\langle (1, \vec{0}) \rangle = \langle (e^{i\theta}, \vec{0}) | e^{i\theta} \in U(1) \rangle \quad (2.46)$$

of all elements connected to $(1, \vec{0})$ by the $U(1)$ action (2.41). Its orbit under $SU(N+1)$ is $\mathbb{CP}^N$. The stability group of (2.46) is

$$S[U(1) \times U(N)] = \left[v \in SU(N+1) : v = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & \hat{v} \end{pmatrix} \right]. \quad (2.47)$$

Thus

$$\mathbb{CP}^N = SU(N+1)/S[U(1) \times U(N)]. \quad (2.48)$$

$S[U(1) \times U(N)]$ is commonly denoted as $U(N)$. The two groups are isomorphic.

To obtain $\mathbb{CP}_F^N$, we think of S^{2N+1} as a submanifold of $\mathbb{C}^{N+1} \setminus \{0\}$:

$$S^{2N+1} = \left\langle \xi = \frac{z}{|z|} : z = (z_1, z_2, \dots, z_{N+1}) \in \mathbb{C}^{N+1} \setminus \{0\} \right\rangle. \quad (2.49)$$

Just as before, we can quantize $\mathbb{C}^{N+1}$ by replacing z_i by annihilation operators a_i and z_i^* by $a_i^\dagger$:

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0, \quad [a_i, a_j^\dagger] = \delta_{ij}. \quad (2.50)$$

With

$$\hat{N} = a_i^\dagger a_i \quad (2.51)$$

as the number operator, the quantized ξ is given by the correspondence

$$\xi_i = \frac{z_i}{|z|} \longrightarrow a_i \frac{1}{\sqrt{\hat{N}}}, \quad N \neq 0. \quad (2.52)$$

Then as in (2.6), we get the $\mathbb{CP}_F^N$ coordinates

$$X_i(z) \longrightarrow X_i = \frac{1}{\hat{N}} a_i^\dagger \lambda_i a, \quad N \neq 0. \quad (2.53)$$

The rest of the discussion follows that of $\mathbb{CP}^1$ with $SU(N+1)$ replacing $SU(2)$. Because of (the analogue of) (2.7), X_i can be restricted to $\mathcal{H}_n$, the subspace of the Fock space with $\hat{N} = n$. It is spanned by the orthonormal vectors

$$\prod_{i=1}^{N+1} \frac{(a_i^\dagger)^{n_i}}{\sqrt{n_i!}} |0\rangle := |n_1, n_2, \dots, N+1\rangle, \quad \sum n_i = n, \quad (2.54)$$

and is of dimension

$$M = {}^{N+1+n}C_n = \frac{(N+n)!}{n!N!}. \quad (2.55)$$

The $SU(N+1)$ angular momentum operators are given by a generalized Schwinger construction :

$$L_i = a^\dagger \frac{\lambda_i}{2} a, \quad [L_i, L_j] = i f_{ijk} L_k. \quad (2.56)$$

$a_i^\dagger$ transform by the unitary irreducible representation (UIR) of dimension $(N+1)$ of $SU(N+1)$ and (2.54) span the space of n fold symmetric product of these UIR's. It carries a UIR of dimension (2.55) and has the quadratic Casimir operator

$$\sum L_i^2 = \frac{N}{2} \left(\frac{n^2}{N+1} + n \right) \mathbf{1}. \quad (2.57)$$

Its remaining Casimir operators are fixed by (2.57). As before

$$\begin{aligned} X_i|_{\mathcal{H}_n} &= \frac{2}{n} L_i|_{\mathcal{H}_n}, \\ [X_i, X_j]|_{\mathcal{H}_n} &= \frac{2}{n} i f_{ijk} X_k|_{\mathcal{H}_n}, \\ (\sum X_i^2)|_{\mathcal{H}_n} &= \left(\frac{2N}{N+1} + \frac{2N}{n} \right) \mathbf{1}|_{\mathcal{H}_n}. \end{aligned} \quad (2.58)$$

The “size” of $\mathbb{CP}_F^N$ is measured by the “radius” $\sqrt{\left(\frac{2N}{N+1} + \frac{2N}{n} \right)}$. In the $n \rightarrow \infty$ limit, the X_i 's also commute and generate $C^\infty(\mathbb{CP}^N)$.

The “wave functions” of $\mathbb{CP}_F^N$ are polynomials in X_i , that is they are elements of $Mat(M)$, with a scalar product like (2.13). As before, for each $\alpha \in Mat(M)$, we have two observables $\alpha_{L,R}$ and they constitute the matrix algebras $Mat_{L,R}(M)$.

The discussions leading up to (2.18) and (2.20) can be adapted also to $\mathbb{CP}_F^N$. As for (2.21), it generalizes to

$$\mathcal{L}_i \longrightarrow -i f_{ijk} X_j(\xi) \frac{\partial}{\partial X_k(\xi)}. \quad (2.59)$$

Diagonalization of $\mathcal{L}_i$ involves the reduction of the product of the UIR's of $SU(N+1)$ given by L_i^L and its complex conjugate given by L_i^R to their irreducible components. The corresponding polarization operators can also in principle be constructed.

The scalar field action (2.28) generalizes easily to $\mathbb{C}P_F^N$.

2.9 The $\mathbb{C}P^N$ Holstein-Primakoff Construction

The generalization of this construction to $\mathbb{C}P^N$ and $SU(N+1)$ is due to Sen [48].

Consider for specificity $N = 2$ and $SU(3)$ first. $SU(3)$ has 3 oscillators a_1, a_2, a_3 . There are also the $SU(2)$ algebras with generators

$$\sum_{i,j=1}^2 a_i^\dagger \left(\frac{\vec{\sigma}}{2} \right)_{ij} a_j, \quad \sum_{i,j=2}^3 a_i^\dagger \left(\frac{\vec{\sigma}}{2} \right)_{ij} a_j, \quad (2.60)$$

acting on the indices 1, 2 and 2, 3 respectively, of a 's and $a^\dagger$'s. Taking their commutators, we can generate the full $SU(3)$ Lie algebra.

We will eliminate $a_2, a_2^\dagger$ from both these sets using the previous Holstein-Primakoff construction. In that way, we will obtain the $SU(3)$ Holstein-Primakoff construction.

As previously we write the polar decompositions

$$a_2 = U_2 \sqrt{N_2}, \quad a_2^\dagger = \sqrt{N_2} U_2^\dagger, \quad N_2 = a_2^\dagger a_2, \quad U_2^\dagger U_2 = \mathbf{1}. \quad (2.61)$$

The oscillators act on the Fock space $\oplus_N \mathcal{H}_N$ spanned by (2.54) for $N = 2$. The actions of U_2 and

$$A_{12}^\dagger = a_1^\dagger U_2, \quad A_{12} = U_2^\dagger a_1 \quad (2.62)$$

follow (2.35) and (2.36). They do not affect n_3 . Using (2.39), we can write the $SU(2)$ generators acting on (12) indices as

$$\begin{aligned} I_+ &= a_1^\dagger a_2 = A_{12}^\dagger \sqrt{N_2}, \\ I_- &= a_2^\dagger a_1 = \sqrt{N_2} A_{12}, \\ I_3 &= \frac{1}{2} (a_1^\dagger a_1 - a_2^\dagger a_2) = \frac{1}{2} (A_{12}^\dagger A_{12} - N_2). \end{aligned} \quad (2.63)$$

We follow the I, U, V spin notation of $SU(3)$ in particle physics [49]. They are connected by Weyl reflections.

In a similar manner, the $SU(3)$ generators acting on (23) indices are constructed from

$$A_{32}^\dagger = a_3^\dagger U_2, \quad A_{32} = U_2^\dagger a_3, \quad (2.64)$$

and read

$$\begin{aligned} U_+ &= a_3^\dagger a_2 = A_{32}^\dagger \sqrt{N_2}, \\ U_- &= a_2^\dagger a_3 = \sqrt{N_2} A_{32}, \\ U_3 &= \frac{1}{2}(a_2^\dagger a_2 - a_3^\dagger a_3) = \frac{1}{2}(N_2 - A_{32}^\dagger A_{32}). \end{aligned} \quad (2.65)$$

In a UIR of $SU(3)$, the total number operator $N = N_1 + N_2 + N_3$ is fixed. Acting on $\mathcal{H}_n$, it becomes n . Keeping this in mind, we now substitute

$$N_2 = N - N_1 - N_3 = N - A_{12}^\dagger A_{12} - A_{32}^\dagger A_{32} \quad (2.66)$$

in (2.63) and (2.65) to eliminate the second oscillator. That gives

$$\begin{aligned} I_+ &= A_{12}^\dagger \sqrt{N - N_1 - N_3}, \\ I_- &= \sqrt{N - N_1 - N_3} A_{12}, \\ I_3 &= N_1 + \frac{N_3}{2} - \frac{N}{2} \end{aligned} \quad (2.67)$$

$$\begin{aligned} U_+ &= A_{32}^\dagger \sqrt{N - N_1 - N_3}, \\ U_- &= \sqrt{N - N_1 - N_3} A_{32}, \\ U_3 &= -\left(N_3 + \frac{N_1}{2} - \frac{N}{2}\right) \end{aligned} \quad (2.68)$$

These operators and their commutators generate the full $SU(3)$ Lie algebra when restricted to $\mathcal{H}_n$. That is the $SU(3)$ Holstein-Primakoff construction.

If the restriction to $\mathcal{H}_n$ is not made, N is a new operator and we get instead the $U(3)$ Lie algebra with N generating its central $U(1)$.

The Holstein-Primakoff construction for $\mathbb{C}P^N$ is much the same. One introduces $N+1$ oscillators $a_i, a_i^\dagger (i \in [1, \dots, N])$ with which the $SU(N+1)$ Lie algebra can be realized using the Schwinger construction. The $SU(N+1)$ UIR's we get therefrom are symmetric products of the fundamental representation of dimension $(N+1)$. The number operator $N = a^\dagger \cdot a$ has a fixed value in one such UIR. Next $a_2, a_2^\dagger$ are eliminated from $SU(N+1)$ generators in favor of the N and the remaining operators to obtain the generalized Holstein-Primakoff construction.

$SU(N+1)$ is of rank N , and we can realize its Lie algebra with N oscillators. There is a similar result in quantum field theory where with the help of the vertex operator construction, a (simply laced) rank N Lie algebra can be realized with N scalar fields on $S^1 \times \mathbb{R}$ valued on S^1 [50]. This resemblance perhaps is not an accident.

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Chapter 3

Star Products

3.1 Introduction

The algebra of smooth functions on a manifold $\mathcal{M}$ under point-wise multiplication is commutative. In deformation quantization [51], this point-wise product is deformed to a non-commutative (but still associative) product called the $*$ -product. It has a central role in many discussions of non-commutative geometry. It has been fruitfully used in quantum optics for a long time.

The existence of such deformations was understood many years ago by Weyl, Wigner, Groenewold and Moyal [52, 40, 43]. They noted that if there is a linear injection (one-to-one map) ψ of an algebra $\mathcal{A}$ into smooth functions $\mathcal{C}^\infty(\mathcal{M})$ on a manifold $\mathcal{M}$, then the product in $\mathcal{A}$ can be transported to the image $\psi(\mathcal{A})$ of $\mathcal{A}$ in $\mathcal{C}^\infty(\mathcal{M})$ using the map. That is then a $*$ -product.

Let us explain this construction with greater completeness and generality [31]. For concreteness we can consider $\mathcal{A}$ to be an algebra of bounded operators on a Hilbert space closed under the Hermitian conjugation of $*$. It is then an example of a $*$ -algebra.

More generally, $\mathcal{A}$ can be a generic “ $*$ -algebra”, that is an algebra closed under an anti-linear involution:

$$a, b \in \mathcal{A}, \lambda \in \mathbb{C} \Rightarrow a^*, b^* \in \mathcal{A}, (ab)^* = b^*a^*, (\lambda a)^* = \lambda^*a^*. \quad (3.1)$$

A two-sided ideal $\mathcal{A}_0$ of $\mathcal{A}$ is a subalgebra of $\mathcal{A}$ with the property

$$a_0 \in \mathcal{A}_0 \Rightarrow \alpha a_0 \text{ and } a_0 \alpha \in \mathcal{A}_0, \forall \alpha \in \mathcal{A}. \quad (3.2)$$

That is $\mathcal{A}\mathcal{A}_0, \mathcal{A}_0\mathcal{A} \subseteq \mathcal{A}_0$. A two-sided $*$ -ideal $\mathcal{A}_0$ by definition is itself closed under $*$ as well.

An element of the quotient $\mathcal{A}/\mathcal{A}_0$ is the equivalence class

$$\{\alpha + \mathcal{A}_0 \subset \mathcal{A}\} = \{[\alpha + a_0] | a_0 \in \mathcal{A}_0\}. \quad (3.3)$$

If $\mathcal{A}_0$ is a two-sided ideal, $\mathcal{A}/\mathcal{A}_0$ is itself an algebra with the sum and the product

$$\begin{aligned} (\alpha + \mathcal{A}_0) + (\beta + \mathcal{A}_0) &= \alpha + \beta + \mathcal{A}_0, \\ (\alpha + \mathcal{A}_0)(\beta + \mathcal{A}_0) &= \alpha\beta + \mathcal{A}_0. \end{aligned} \quad (3.4)$$

If $\mathcal{A}_0$ is a two-sided $*$ -ideal, then $\mathcal{A}/\mathcal{A}_0$ is a $*$ -algebra with the $*$ -operation

$$(\alpha + \mathcal{A}_0)^* = \alpha^* + \mathcal{A}_0. \quad (3.5)$$

We note that the product and $*$ are independent of the choice of the representatives α, β from the equivalence classes $\alpha + \mathcal{A}_0$ and $\beta + \mathcal{A}_0$ because $\mathcal{A}_0$ is a two-sided ideal. So they make sense for $\mathcal{A}/\mathcal{A}_0$.

Let $C^\infty(\mathcal{M})$ denote the complex-valued smooth functions on a manifold $\mathcal{M}$. Complex conjugation $-(\text{bar})$ is defined on these functions. It sends a function f to its complex conjugate $\bar{f}$.

We consider the linear maps

$$\psi : \mathcal{A} \longrightarrow C^\infty(\mathcal{M}), \quad (3.6)$$

$$\psi\left(\sum \lambda_i a_i\right) = \sum \lambda_i \psi(a_i), \quad a_i \in \mathcal{A}, \quad \lambda_i \in \mathbb{C}. \quad (3.7)$$

The kernel of such a map is the set of all $\alpha \in \mathcal{A}$ for which $\psi(\alpha)$ is the zero function 0. (Its value is zero at all points of $\mathcal{M}$):

$$\text{Ker } \psi = \langle \alpha_0 \in \mathcal{A} | \psi(\alpha_0) = 0 \rangle. \quad (3.8)$$

ψ descends to a linear map, called Ψ , from $\mathcal{A}/\text{Ker } \psi = \{\alpha + \text{Ker } \psi : \alpha \in \mathcal{A}\}$ to $C^\infty(\mathcal{M})$:

$$\Psi(\alpha + \text{Ker } \psi) = \psi(\alpha). \quad (3.9)$$

$\psi(\alpha)$ does not depend on the choice of the representative α from $\alpha + \text{Ker } \psi$ because of (3.8). Clearly Ψ is an injective map from $\mathcal{A}/\text{Ker } \psi$ to $C^\infty(\mathcal{M})$.

If $\text{Ker } \psi$ is also a two-sided ideal, Ψ is a linear map from the algebra $\mathcal{A}/\text{Ker } \psi$ to $C^\infty(\mathcal{M})$. Using this fact, we define a product, also denoted by $*$, on $\Psi(\mathcal{A}/\text{Ker } \psi) = \psi(\mathcal{A}) \subseteq C^\infty(\mathcal{M})$:

$$\Psi(\alpha + \text{Ker } \psi) * \Psi(\beta + \text{Ker } \psi) = \Psi((\alpha + \text{Ker } \psi)(\beta + \text{Ker } \psi)). \quad (3.10)$$

or

$$\psi(\alpha) * \psi(\beta) = \psi(\alpha\beta). \quad (3.11)$$

With this product, $\psi(\mathcal{A})$ is an algebra $(\psi(\mathcal{A}), *)$ isomorphic to $\mathcal{A}/\text{Ker } \psi$. (The notation means that $\psi(\mathcal{A})$ is considered with product $*$ and not say point-wise product).

We assume that $\mathcal{A}/\text{Ker } \psi$ is a $*$ -algebra and that Ψ preserves the stars on $\mathcal{A}/\text{Ker } \psi$ and $C^\infty(\mathcal{M})$, the $*$ on the latter being complex conjugation denoted by bar:

$$\begin{aligned}\Psi((\alpha + \text{Ker } \psi)^*) &= \overline{\Psi(\alpha + \text{Ker } \psi)}, \\ \psi(\alpha^*) &= \overline{\psi(\alpha)}.\end{aligned}\tag{3.12}$$

Such ψ and Ψ are said to be $*$ -morphisms from $\mathcal{A}$ and $\mathcal{A}/\text{Ker } \psi$ to $(\psi(\mathcal{A}), *)$. The two algebras $\mathcal{A}/\text{Ker } \psi$ and $(\psi(\mathcal{A}), *)$ are $*$ -isomorphic.

Remark: Star $(*)$ occurs with two meanings.

- (1) It refers to involution on algebras in the phrase $*$ -morphism.
- (2) It refers to the new product on functions in $(\psi(\mathcal{A}), *)$.

These confusing notations, designed to keep the reader alert, are standard in the literature.

The above is the general framework. In applications, we encounter more than one linear bijection (one-to-one, onto map) from an algebra $\mathcal{A}$ to $C^\infty(\mathcal{M})$ and that produces different-looking $*$'s on $C^\infty(\mathcal{M})$ and algebras $(C^\infty(\mathcal{M}), *)$, $(C^\infty(\mathcal{M}), *')$ etc. As they are $*$ -isomorphic to $\mathcal{A}$, they are mutually $*$ -isomorphic as well. A simple example we encounter below is $C^\infty(\mathbb{C})$ with Moyal- and coherent-state-induced $*$ -products. These algebras are $*$ -isomorphic.

3.2 Properties of Coherent States

It is useful to have the Campbell-Baker-Hausdorff (CBH) formula written down. It reads

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{A} + \hat{B}} e^{\frac{1}{2}[\hat{A}, \hat{B}]}\tag{3.13}$$

for two operators $\hat{A}, \hat{B}$ if

$$[\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0.\tag{3.14}$$

For one oscillator with annihilation-creation operators $a, a^\dagger$, the coherent state

$$|z\rangle = e^{za^\dagger - \bar{z}a}|0\rangle = e^{-\frac{1}{2}|z|^2} e^{za^\dagger}|0\rangle, \quad z \in \mathbb{C}\tag{3.15}$$

has the properties

$$a|z\rangle = z|z\rangle; \quad \langle z'|z\rangle = e^{\frac{1}{2}|z-z'|^2}. \quad (3.16)$$

The coherent states are overcomplete, with the resolution of identity

$$\mathbf{1} = \int \frac{d^2z}{\pi} |z\rangle\langle z|, \quad d^2z = dx_1 dx_2, \quad \text{where} \quad z = \frac{x_1 + ix_2}{\sqrt{2}}. \quad (3.17)$$

The factor $\frac{1}{\pi}$ is easily checked: $\text{Tr} \mathbf{1}|0\rangle\langle 0| = 1$ while $\int d^2z |\langle 0|z\rangle|^2$ is π in view of (3.16).

A central property of coherent states is the following: an operator $\hat{A}$ is determined just by its diagonal matrix elements

$$A(z, \bar{z}) = \langle z|\hat{A}|z\rangle, \quad (3.18)$$

that is by its “symbol” A , a function* on $\mathbb{C}$ with values $A(z, \bar{z}) = \langle z|\hat{A}|z\rangle$. An easy proof uses analyticity [54]. $\hat{A}$ is certainly determined by the collection of all its matrix elements $\langle \bar{\eta}|\hat{A}|\xi\rangle$ or equally by

$$e^{\frac{1}{2}(|\eta|^2 + |\xi|^2)} \langle \bar{\eta}|\hat{A}|\xi\rangle = \langle 0|e^{\eta a} \hat{A} e^{\xi a^\dagger} |0\rangle. \quad (3.19)$$

The right hand side (at least for appropriate $\mathcal{A}$) is seen to be a holomorphic function of η and ξ , or equally well of

$$u = \frac{\eta + \xi}{2}, \quad v = \frac{\eta - \xi}{2i}. \quad (3.20)$$

Holomorphic functions are globally determined by their values for real arguments. Hence the function $\tilde{A}$ defined by

$$\tilde{A}(u, v) = \langle 0|e^{\eta a^\dagger} \hat{A} e^{\xi a} |0\rangle \quad (3.21)$$

is globally determined by its values for u, v real or $\eta = \bar{\xi}$. Thus $\langle \xi|\hat{A}|\xi\rangle$ determines $\hat{A}$ as claimed.

There are also explicit formulas for $\hat{A}$ in terms of $\langle \xi|\bar{A}|\xi\rangle$ [55].

3.3 The Coherent State or Voros *-Product on the Moyal Plane

As indicated above, we can map an operator $\hat{A}$ to a function A using coherent states as follows:

$$\hat{A} \longrightarrow A, \quad A(z, \bar{z}) = \langle z|\hat{A}|z\rangle. \quad (3.22)$$

*The $\bar{z}$ argument in $A(z, \bar{z})$ is redundant. It is there to emphasize that A is not necessarily a holomorphic function of the complex variable z .

This map is linear and also bijective by the previous remarks and induces a product $*_C$ on functions (C indicating “coherent state”). With this product, we get an algebra $(C^\infty(\mathbb{C}), *_C)$ of functions. Since the map $\hat{A} \rightarrow A$ has the property $\hat{A}^* \rightarrow A^* \equiv \bar{A}$, this map is a $*$ -morphism from operators to $(C^\infty(\mathbb{C}), *_C)$.

Let us get familiar with this new function algebra.

The image of a is the function α where $\alpha(z, \bar{z}) = z$. The image of a^n has the value z^n at $(z, \bar{z})$, so by definition,

$$(\alpha *_C \alpha \dots *_C \alpha)(z, \bar{z}) = z^n. \quad (3.23)$$

The image of $a^* \equiv a^\dagger$ is $\bar{\alpha}$ where $\bar{\alpha}(z, \bar{z}) = \bar{z}$ and that of $(a^*)^n$ is $\bar{\alpha} *_C \bar{\alpha} \dots *_C \bar{\alpha}$ where

$$(\bar{\alpha} *_C \bar{\alpha} \dots *_C \bar{\alpha})(z, \bar{z}) = \bar{z}^n. \quad (3.24)$$

Since $\langle z | a^* a | z \rangle = \bar{z}z$ and $\langle z | a a^* | z \rangle = \bar{z}z + 1$, we get

$$\bar{\alpha} *_C \alpha = \bar{\alpha}\alpha, \quad \alpha *_C \bar{\alpha} = \alpha\bar{\alpha} + \mathbf{1}, \quad (3.25)$$

where $\bar{\alpha}\alpha = \alpha\bar{\alpha}$ is the pointwise product of α and $\bar{\alpha}$, and $\mathbf{1}$ is the constant function with value 1 for all z .

For general operators $\hat{f}$, the construction proceeds as follows. Consider

$$: e^{\xi a^\dagger - \bar{\xi} a} : \quad (3.26)$$

where the normal ordering symbol $: \dots :$ means as usual that $a^\dagger$'s are to be put to the left of a 's. Thus

$$\begin{aligned} : a a^\dagger a^\dagger a : &= a^\dagger a^\dagger a a, \\ : e^{\xi a^\dagger - \bar{\xi} a} : &= e^{\xi a^\dagger} e^{-\bar{\xi} a}. \end{aligned} \quad (3.27)$$

Hence

$$\langle z | : e^{\xi a^\dagger - \bar{\xi} a} : | z \rangle = e^{\xi \bar{z} - \bar{\xi} z}. \quad (3.28)$$

Writing $\hat{f}$ as a Fourier transform,

$$\hat{f} = \int \frac{d^2 \xi}{\pi} : e^{\xi a^\dagger - \bar{\xi} a} : \tilde{f}(\xi, \bar{\xi}), \quad \tilde{f}(\xi, \bar{\xi}) \in \mathbb{C}, \quad (3.29)$$

its symbol is seen to be

$$f = \int \frac{d^2 \xi}{\pi} e^{\xi \bar{z} - \bar{\xi} z} \tilde{f}(\xi, \bar{\xi}). \quad (3.30)$$

This map is invertible since f determines $\tilde{f}$.

Consider also the second operator

$$\hat{g} = \int \frac{d^2\eta}{\pi} : e^{\eta a^\dagger - \bar{\eta} a} : \tilde{g}(\eta, \bar{\eta}), \quad (3.31)$$

and its symbol

$$g = \int \frac{d^2\eta}{\pi} e^{\eta \bar{z} - \bar{\eta} z} \tilde{g}(\eta, \bar{\eta}). \quad (3.32)$$

The task is to find the symbol $f *_C g$ of $\hat{f}\hat{g}$.

Let us first find

$$e^{\xi \bar{z} - \bar{\xi} z} *_C e^{\eta \bar{z} - \bar{\eta} z}. \quad (3.33)$$

We have

$$: e^{\xi a^\dagger - \bar{\xi} a} :: e^{\eta a^\dagger - \bar{\eta} a} :=: e^{\xi a^\dagger - \bar{\xi} a} e^{\eta a^\dagger - \bar{\eta} a} : e^{-\bar{\xi} \eta} \quad (3.34)$$

and hence

$$\begin{aligned} e^{\xi \bar{z} - \bar{\xi} z} *_C e^{\eta \bar{z} - \bar{\eta} z} &= e^{-\bar{\xi} \eta} e^{\xi \bar{z} - \bar{\xi} z} e^{\eta \bar{z} - \bar{\eta} z} \\ &= e^{\xi \bar{z} - \bar{\xi} z} e^{\overleftarrow{\partial}_z \overrightarrow{\partial}_{\bar{z}}} e^{\eta \bar{z} - \bar{\eta} z}. \end{aligned} \quad (3.35)$$

The bidifferential operators $(\overleftarrow{\partial}_z \overrightarrow{\partial}_{\bar{z}})^k$, $(k = 1, 2, \dots)$ have the definition

$$\alpha(\overleftarrow{\partial}_z \overrightarrow{\partial}_{\bar{z}})^k \beta(z, \bar{z}) = \frac{\partial^k \alpha(z, \bar{z})}{\partial z^k} \frac{\partial^k \beta(z, \bar{z})}{\partial \bar{z}^k}. \quad (3.36)$$

The exponential in (3.35) involving them can be defined using the power series.

$f *_C g$ follows from (3.35):

$$(f *_C g)(z, \bar{z}) = (f e^{\overleftarrow{\partial}_z \overrightarrow{\partial}_{\bar{z}}} g)(z, \bar{z}). \quad (3.37)$$

(3.37) is the coherent state $*$ -product [56].

We can explicitly introduce a deformation parameter $\theta > 0$ in the discussion by changing (3.37) to

$$f *_C g(z, \bar{z}) = (f e^{\theta \overleftarrow{\partial}_z \overrightarrow{\partial}_{\bar{z}}} g)(z, \bar{z}). \quad (3.38)$$

After rescaling $z' = \frac{z}{\sqrt{\theta}}$, (3.38) gives (3.37). As z' and $\bar{z}'$ after quantization become $a, a^\dagger$, z and $\bar{z}$ become the scaled oscillators $a_\theta, a_\theta^\dagger$:

$$[a_\theta, a_\theta] = [a_\theta^\dagger, a_\theta^\dagger] = 0, \quad [a_\theta, a_\theta^\dagger] = \theta. \quad (3.39)$$

(3.39) is associated with the Moyal plane with Cartesian coordinate functions x_1, x_2 . If $a_\theta = \frac{x_1 + ix_2}{\sqrt{2}}$, $a_\theta^\dagger = \frac{x_1 - ix_2}{\sqrt{2}}$,

$$[x_i, x_j] = i\theta \varepsilon_{ij}, \quad \varepsilon_{ij} = -\varepsilon_{ji}, \quad \varepsilon_{12} = 1. \quad (3.40)$$

The Moyal plane is the plane $\mathbb{R}^2$, but with its function algebra deformed in accordance with (3.40). The deformed algebra has the product (3.38) or equivalently the Moyal product derived below.

Discussion of the equivalence between Moyal and Voros star products, using Stratonovich's methods [57] and with several physical applications is given in [58, 59].

3.4 The Moyal $\ast$ -Product on the Groenewold-Moyal Plane

We get this by changing the map $\hat{f} \rightarrow f$ from operators to functions. For a given function f , the operator $\hat{f}$ is thus different for the coherent state and Moyal $\ast$'s. The $\ast$ -product on two functions is accordingly also different.

3.4.1 The Weyl Map and the Weyl Symbol

The Weyl map of the operator

$$\hat{f} = \int \frac{d^2\xi}{\pi} \tilde{f}(\xi, \bar{\xi}) e^{\xi a^\dagger - \bar{\xi} a}, \quad (3.41)$$

to the function f is defined by

$$f(z, \bar{z}) = \int \frac{d^2\xi}{\pi} \tilde{f}(\xi, \bar{\xi}) e^{\xi \bar{z} - \bar{\xi} z}. \quad (3.42)$$

(3.42) makes sense since $\tilde{f}$ is fully determined by $\hat{f}$ as follows:

$$\langle z | \hat{f} | z \rangle = \int \frac{d^2\xi}{\pi} \tilde{f}(\xi, \bar{\xi}) e^{-\frac{1}{2}\xi\bar{\xi}} e^{\xi \bar{z} - \bar{\xi} z}. \quad (3.43)$$

$\tilde{f}$ can be calculated from here by Fourier transformation.

The map is invertible since $\tilde{f}$ follows from f by Fourier transform of (3.42) and $\tilde{f}$ fixes $\hat{f}$ by (3.41).

f is called the *Weyl symbol* of $\hat{f}$.

As the Weyl map is bijective, we can find a new $\ast$ product, call it $\ast_W$, between functions by setting $f \ast_W g =$ Weyl symbol of $\hat{f}\hat{g}$.

For

$$\hat{f} = e^{\xi a^\dagger - \bar{\xi} a}, \quad \hat{g} = e^{\eta a^\dagger - \bar{\eta} a}, \quad (3.44)$$

to find $f \ast_W g$, we first rewrite $\hat{f}\hat{g}$ according to

$$\hat{f}\hat{g} = e^{\frac{1}{2}(\xi\bar{\eta} - \bar{\xi}\eta)} e^{(\xi+\eta)a^\dagger - (\bar{\xi}+\bar{\eta})a}. \quad (3.45)$$

Hence

$$\begin{aligned} (f \ast_W g)(z, \bar{z}) &= e^{\xi \bar{z} - \bar{\xi} z} e^{\frac{1}{2}(\xi\bar{\eta} - \bar{\xi}\eta)} e^{\eta \bar{z} - \bar{\eta} z} \\ &= f e^{\frac{1}{2}(\overleftarrow{\partial}_z \overrightarrow{\partial}_{\bar{z}} - \overleftarrow{\partial}_{\bar{z}} \overrightarrow{\partial}_z)} g(z, \bar{z}). \end{aligned} \quad (3.46)$$

Multiplying by $\tilde{f}(\xi, \bar{\xi})$, $\tilde{g}(\eta, \bar{\eta})$ and integrating, we get (3.46) for arbitrary functions:

$$(f \ast_W g)(z, \bar{z}) = \left(f e^{\frac{1}{2}(\overleftarrow{\partial}_z \overrightarrow{\partial}_{\bar{z}} - \overleftarrow{\partial}_{\bar{z}} \overrightarrow{\partial}_z)} g \right)(z, \bar{z}). \quad (3.47)$$

Note that

$$\overleftarrow{\partial}_z \overrightarrow{\partial}_{\bar{z}} - \overleftarrow{\partial}_{\bar{z}} \overrightarrow{\partial}_z = i(\overleftarrow{\partial}_1 \overrightarrow{\partial}_2 - \overleftarrow{\partial}_2 \overrightarrow{\partial}_1) = i\varepsilon_{ij} \overleftarrow{\partial}_i \overrightarrow{\partial}_j. \quad (3.48)$$

Introducing also θ , we can write the $*_W$ -product as

$$f *_W g = f e^{i\frac{\theta}{2}\varepsilon_{ij} \overleftarrow{\partial}_i \overrightarrow{\partial}_j} g. \quad (3.49)$$

By (3.40), $\theta\varepsilon_{ij} = \omega_{ij}$ fixes the Poisson brackets, or the Poisson structure on the Moyal plane. (3.49) is customarily written as

$$f *_W g = f e^{\frac{i}{2}\omega_{ij} \overleftarrow{\partial}_i \overrightarrow{\partial}_j} g. \quad (3.50)$$

using the Poisson structure. (But we have not cared to position the indices so as to indicate their tensor nature and to write ω^{ij} .)

3.5 Properties of $*$ -Products

A $*$ -product without a subscript indicates that it can be either a $*_C$ or a $*_W$.

3.5.1 Cyclic Invariance

The trace of operators has the fundamental property

$$Tr \hat{A} \hat{B} = Tr \hat{B} \hat{A} \quad (3.51)$$

which leads to the general cyclic identities

$$Tr \hat{A}_1 \dots \hat{A}_n = Tr \hat{A}_n \hat{A}_1 \dots \hat{A}_{n-1}. \quad (3.52)$$

We now show that

$$Tr \hat{A} \hat{B} = \int \frac{d^2 z}{\pi} A * B(z, \bar{z}), \quad * = *_C \quad \text{or} \quad *_W. \quad (3.53)$$

(The functions on R.H.S. are different for $*_C$ and $*_W$ if $\hat{A}, \hat{B}$ are fixed). From this follows the analogue of (3.52):

$$\int \frac{d^2 z}{\pi} (A_1 * A_2 * \dots * A_n)(z, \bar{z}) = \int \frac{d^2 z}{\pi} (A_n * A_1 * \dots * A_{n-1})(z, \bar{z}). \quad (3.54)$$

For $*_C$, (3.53) follows from (3.17).

The coherent state image of $e^{\xi a^\dagger - \bar{\xi} a}$ is the function with value

$$e^{\xi \bar{z} - \bar{\xi} z} e^{-\frac{1}{2} \xi \bar{\xi}} \quad (3.55)$$

at z , with a similar correspondence if $\xi \rightarrow \eta$. So

$$Tr e^{\xi a^\dagger - \bar{\xi} a} e^{\eta a^\dagger - \bar{\eta} a} = \int \frac{d^2 z}{\pi} \left(e^{\xi \bar{z} - \bar{\xi} z} e^{-\frac{1}{2} \bar{\xi} \xi} \right) \left(e^{\eta \bar{z} - \bar{\eta} z} e^{-\frac{1}{2} \bar{\eta} \eta} \right) e^{-\bar{\xi} \eta} \quad (3.56)$$

The integral produces the δ -function

$$\prod_i 2\delta(\xi_i + \eta_i), \quad \xi_i = \frac{\xi_1 + \xi_2}{\sqrt{2}}, \quad \eta_i = \frac{\eta_1 + \eta_2}{\sqrt{2}}. \quad (3.57)$$

We can hence substitute $e^{-(\frac{1}{2} \bar{\xi} \xi + \frac{1}{2} \bar{\eta} \eta + \bar{\xi} \eta)}$ by $e^{\frac{1}{2}(\xi \bar{\eta} - \bar{\xi} \eta)}$ and get (3.53) for Weyl $*$ for these exponentials and so for general functions by using (3.41).

3.5.2 A Special Identity for the Weyl Star

The above calculation also gives, the identity

$$\int \frac{d^2 z}{\pi} (A *_W B)(z, \bar{z}) = \int \frac{d^2 z}{\pi} A(z, \bar{z}) B(z, \bar{z}). \quad (3.58)$$

That is because

$$\prod_i \delta(\xi_i + \eta_i) e^{\frac{1}{2}(\xi \bar{\eta} - \bar{\xi} \eta)} = \prod_i \delta(\xi_i + \eta_i). \quad (3.59)$$

In (3.54), A and B in turn can be Weyl $*$ -products of other functions. Thus in integrals of Weyl $*$ -products of functions, one $*$ $_W$ can be replaced by the pointwise (commutative) product:

$$\begin{aligned} & \int \frac{d^2 z}{\pi} (A_1 *_W A_2 *_W \cdots A_K) *_W (B_1 *_W B_2 *_W \cdots B_L)(z, \bar{z}) \\ &= \int \frac{d^2 z}{\pi} (A_1 *_W A_2 *_W \cdots A_K) (B_1 *_W B_2 *_W \cdots B_L)(z, \bar{z}). \end{aligned} \quad (3.60)$$

This identity is frequently useful.

3.5.3 Equivalence of $*_C$ and $*_W$

For the operator

$$\hat{A} = e^{\xi a^\dagger - \bar{\xi} a}, \quad (3.61)$$

the coherent state function A_C has the value (3.55) at z , and the Weyl symbol A_W has the value

$$A_W(z, \bar{z}) = e^{\xi \bar{z} - \bar{\xi} z}. \quad (3.62)$$

As both $(C^\infty(\mathbb{R}^2), *_C)$ and $(C^\infty(\mathbb{R}^2), *_W)$ are isomorphic to the operator algebra, they too are isomorphic. The isomorphism is established by the maps

$$A_C \longleftrightarrow A_W \quad (3.63)$$

and their extension via Fourier transform to all operators $\hat{A}$ and functions $A_{C,W}$.

Clearly

$$A_W = e^{-\frac{1}{2}\partial_z\partial_{\bar{z}}} A_C, \quad A_C = e^{\frac{1}{2}\partial_z\partial_{\bar{z}}} A_W, \\ A_C *_C B_C \longleftrightarrow A_W *_W B_W. \quad (3.64)$$

The mutual isomorphism of these three algebras is a $*$ -isomorphism since $(\hat{A}\hat{B})^\dagger \longrightarrow \bar{B}_{C,W} *_C \bar{A}_{C,W}$.

3.5.4 Integration and Tracial States

This is a good point to introduce the ideas of a state and a tracial state on a $*$ -algebra $\mathcal{A}$ with unity **1**.

A state ω is a linear map from $\mathcal{A}$ to $\mathbb{C}$, $\omega(a) \in \mathbb{C}$ for all $a \in \mathcal{A}$ with the following properties:

$$\omega(a^*) = \overline{\omega(a)}, \\ \omega(a^*a) \geq 0, \\ \omega(\mathbf{1}) = 1. \quad (3.65)$$

If $\mathcal{A}$ consists of operators on a Hilbert space and ρ is a density matrix, it defines a state ω_ρ via

$$\omega_\rho(a) = \text{Tr}(\rho a). \quad (3.66)$$

If $\rho = e^{-\beta H} / \text{Tr}(e^{-\beta H})$ for a Hamiltonian H , it gives a Gibbs state via (3.66).

Thus the concept of a state on an algebra $\mathcal{A}$ generalizes the notion of a density matrix. There is a remarkable construction, the Gel'fand- Naimark-Segal (GNS) construction which shows how to associate any state with a rank-1 density matrix [60].

A state is *tracial* if it has cyclic invariance [60]:

$$\omega(ab) = \omega(ba). \quad (3.67)$$

The Gibbs state is not tracial, but fulfills an identity generalizing (3.67). It is a Kubo-Martin-Schwinger (KMS) state [60].

A positive map ω' is in general an unnormalized state: It must fulfill all the conditions that a state fulfills, but is not obliged to fulfill the condition $\omega'(\mathbf{1}) = 1$.

Let us define a positive map ω' on $(C^\infty(\mathbb{R}^2), *)$ ($*$ = $*_C$ or $*_W$) using integration:

$$\omega'(A) = \int \frac{d^2 z}{\pi} \hat{A}(z, \bar{z}). \quad (3.68)$$

It is easy to verify that ω' fulfills the properties of a positive map.

A *tracial* positive map ω' also has the cyclic invariance (3.67).

The cyclic invariance (3.67) of $\omega'(A*B)$ means that it is a tracial positive map.

3.5.5 The θ -Expansion

On introducing θ , we have (3.38) and

$$f *_W g(z, \bar{z}) = f e^{\frac{\theta}{2} (\overleftarrow{\partial}_z \overrightarrow{\partial}_{\bar{z}} - \overleftarrow{\partial}_{\bar{z}} \overrightarrow{\partial}_z)} g(z, \bar{z}). \quad (3.69)$$

The series expansion in θ is thus

$$f *_C g(z, \bar{z}) = fg(z, \bar{z}) + \theta \frac{\partial f}{\partial z}(z, \bar{z}) \frac{\partial g}{\partial \bar{z}}(z, \bar{z}) + \mathcal{O}(\theta^2), \quad (3.70)$$

$$f *_W g(z, \bar{z}) = fg(z, \bar{z}) + \frac{\theta}{2} \left(\frac{\partial f}{\partial z} \frac{\partial g}{\partial \bar{z}} - \frac{\partial f}{\partial \bar{z}} \frac{\partial g}{\partial z} \right) (z, \bar{z}) + \mathcal{O}(\theta^2). \quad (3.71)$$

Introducing the notation

$$[f, g]_* = f *_C g - g *_C f, \quad * = *_C \quad \text{or} \quad *_W, \quad (3.72)$$

We see that

$$\begin{aligned} [f, g]_{*_C} &= \theta \left(\frac{\partial f}{\partial z} \frac{\partial g}{\partial \bar{z}} - \frac{\partial f}{\partial \bar{z}} \frac{\partial g}{\partial z} \right) (z, \bar{z}) + \mathcal{O}(\theta^2), \\ [f, g]_{*_W} &= \theta \left(\frac{\partial f}{\partial z} \frac{\partial g}{\partial \bar{z}} - \frac{\partial f}{\partial \bar{z}} \frac{\partial g}{\partial z} \right) (z, \bar{z}) + \mathcal{O}(\theta^2), \end{aligned} \quad (3.73)$$

We thus see that

$$[f, g]_* = i\theta \{f, g\}_{P.B.} + \mathcal{O}(\theta^2), \quad (3.74)$$

where $\{f, g\}$ is the Poisson Bracket of f and g and the $\mathcal{O}(\theta^2)$ term depends on $*_C, *_W$. Thus the $*$ -product is an associative product which to leading order in the deformation parameter ("Planck's" constant) θ is compatible with the rules of quantization of Dirac. We can say that with the $*$ -product,

we have deformation quantization of the classical commutative algebra of functions.

But it should be emphasized that even to leading order in θ , $f *_C g$ and $f *_W g$ do not agree. Still the algebras $(C^\infty(\mathbb{R}^2, *_C))$ and $(C^\infty(\mathbb{R}^2, *_W))$ are $*$ -isomorphic.

Suppose we are given a Poisson structure on a manifold M with Poisson bracket $\{.,.\}$. Then Kontsevich ([61]) has given a $*$ -product $f * g$ as a formal power series in θ such that (3.74) holds.

For further developments on the physical equivalence of $*_C$ and $*_W$ see, [62].

3.6 The $*$ -Products for the Fuzzy Sphere

Star products for Kähler manifolds have been known for a long time. The approach we take here was initiated by Prešnajder [63] (See also [62], it produces particularly compact expressions.).

Let P_n be the orthogonal projection operator to the subspace with $N = n$. The fuzzy sphere algebra is then the algebra with elements $P_n \gamma(a_i^\dagger a_j) P_n$ where γ is any polynomial in $a_i^\dagger a_j$. As any such polynomial commutes with N , if γ and δ are two of these polynomials,

$$P_n \gamma(a_i^\dagger a_j) P_n P_n \delta(a_i^\dagger a_j) P_n = P_n \gamma(a_i^\dagger a_j) \delta(a_i^\dagger a_j) P_n. \quad (3.75)$$

This algebra, more precisely, is the orthogonal direct sum $Mat(n+1) \oplus 0$ where $Mat(n+1)$ acts on the $\hat{N} = n$ subspace and is the fuzzy sphere. But the extra 0 here is entirely harmless.

3.6.1 The Coherent State $*$ -Product $*_C$

There are now two oscillators a_1, a_2 , so the coherent states are labeled by two complex variables, being

$$|Z_1, Z_2\rangle = e^{Z a^\dagger - \bar{Z} a} |0\rangle, \quad Z = (Z_1, Z_2). \quad (3.76)$$

We use capital Z 's for unnormalized Z 's and z 's for normalized ones: $z = \frac{Z}{|Z|}$, $|Z|^2 = \sum |Z_i|^2$.

The normalized coherent states $|z\rangle_n$ for S_F^2 , as one can guess, are obtained by projection from $|Z\rangle$,

$$|z\rangle_n = \frac{P_n |Z\rangle}{|\langle P_n | Z \rangle|} = \frac{(\sum_i z_i a_i^\dagger)^n}{\sqrt{n!}} |0\rangle, \quad (3.77)$$

where we have used

$$P_n|Z\rangle = \frac{(Z_i a_i^\dagger)^n}{n!} |0\rangle. \quad (3.78)$$

They are called Perelomov coherent states [54]

For an operator $P_n \hat{A} P_n$, the coherent state symbol has the value

$$\langle Z | P_n \hat{A} P_n | Z \rangle = \frac{|z|^{2n}}{n!} \langle z | \hat{A} | z \rangle_n \quad (3.79)$$

at Z . By a previous result, the diagonal coherent state expectation values $\langle z | P_n \hat{A} P_n | z \rangle_n$ determines $P_n \hat{A} P_n$ uniquely and there is a $*$ -product for S_F^2 . We call it a $*_C$ -product in analogy to the notation used before.

We can find it explicitly as follows [63, 31, 9]. For $n = 1$ (spin $\frac{n}{2} = \frac{1}{2}$), a basis for 2×2 matrices is

$$\{\sigma_A : \sigma_0 = \mathbf{1}, \sigma_i \ (i = 1, 2, 3) = \text{Pauli Matrices}, \quad \text{Tr} \sigma_A \sigma_B = 2\delta_{AB}\}. \quad (3.80)$$

Let

$$|i\rangle = a_i^\dagger |0\rangle, \quad i = 1, 2 \quad (3.81)$$

be a pair of orthonormal vectors for $n = 1$. A general operator is

$$\hat{F} = f_A \hat{\sigma}_A, \quad \hat{\sigma}_A = a^\dagger \sigma_A a|_{n=1}, \quad f_A \in \mathbb{C}. \quad (3.82)$$

and $\hat{\sigma}_A |i\rangle = |j\rangle (\sigma_A)_{ji}$. In above, by $a^\dagger \sigma_A a|_{n=1}$, we mean the restriction of $a^\dagger \sigma_A a$ to the subspace with $n = 1$.

Call the coherent state symbol of $\hat{\sigma}_A$ for $n = 1$ as χ_A :

$$\chi_A(z) = \langle z | \hat{\sigma}_A | z \rangle, \quad \chi_0(z) = 1, \quad \chi_i = \bar{z} \sigma_i z, \quad i = 1, 2, 3. \quad (3.83)$$

The $*$ -product for $n = 1$ now follows:

$$\chi_A *_C \chi_B(z) = \langle z | \hat{\sigma}_A \hat{\sigma}_B | z \rangle. \quad (3.84)$$

Write

$$\sigma_A \sigma_B = \delta_{AB} + E_{ABi} \sigma_i \quad (3.85)$$

to get

$$\begin{aligned} \chi_A *_C \chi_B(z) &= \delta_{AB} + E_{ABi} \chi_i(z) \\ &:= \chi_A(z) \chi_B(z) + \mathcal{K}_{AB}(z). \end{aligned} \quad (3.86)$$

Let us use the notation

$$n_i = \chi_i(z), \quad n_0 = 1. \quad (3.87)$$

$\vec{n}$ is the coordinate on S^2 : $\vec{n} \cdot \vec{n} = 1$. Then (3.86) is

$$n_A * n_B(z) = n_A n_B + K_{AB}(n), \quad \mathcal{K}_{AB}(z) := K_{AB}(n). \quad (3.88)$$

This K_{AB} has a particular significance for complex analysis. Since $\chi_0(z) = 1$, $\chi_0(z) * \chi_A = \chi_0 \chi_A$ by (3.86) and

$$K_{0A} = 0. \quad (3.89)$$

The components $K_{ij}(n)$ of K can be calculated from (3.85), (3.86). Let $\theta(\alpha)$ be the spin 1 angular momentum matrices:

$$\theta(\alpha)_{ij} = -i\varepsilon_{\alpha ij}. \quad (3.90)$$

Then

$$K_{ij}(\vec{n}) = \frac{\{\vec{\theta} \cdot \vec{n} (\vec{\theta} \cdot \vec{n} - 1)\}_{ij}}{2}$$

$$\vec{\theta} \cdot \vec{n} := \theta(\alpha) n_\alpha. \quad (3.91)$$

The eigenvalues of $\vec{\theta} \cdot \vec{n}$ are $\pm 1, 0$ and $K_{ij}(\vec{n})$ is the projection operator to the eigenspace $\vec{\theta} \cdot \vec{n} = -1$,

$$K(\vec{n})^2 = K(\vec{n}). \quad (3.92)$$

It is related to the complex structure of S^2 in the projective module picture treated in chapter 5.

The vector space for angular momentum $\frac{n}{2}$ is the n -fold symmetric tensor product of the spin- $\frac{1}{2}$ vector spaces. The general linear operator on this space can be written as

$$\hat{F} = f_{A_1 A_2 \dots A_n} \hat{\sigma}_{A_1} \otimes \hat{\sigma}_{A_2} \otimes \dots \otimes \hat{\sigma}_{A_n} \quad (3.93)$$

where f is totally symmetric in its indices. Its symbol is thus

$$F(\vec{n}) = f_{A_1 A_2 \dots A_n} n_{A_1} n_{A_2} \dots n_{A_n}, \quad n_0 := 1. \quad (3.94)$$

The symbol of another operator

$$\hat{G} = g_{B_1 B_2 \dots B_n} \hat{\sigma}_{B_1} \otimes \hat{\sigma}_{B_2} \otimes \dots \otimes \hat{\sigma}_{B_n}, \quad (3.95)$$

where g is symmetric in its indices, is

$$G(\vec{n}) = g_{B_1 B_2 \dots B_n} n_{B_1} n_{B_2} \dots n_{B_n}. \quad (3.96)$$

Since

$$\hat{F}\hat{G} = f_{A_1 A_2 \dots A_n} g_{B_1 B_2 \dots B_n} \sigma_{A_1} \sigma_{B_1} \otimes \sigma_{A_2} \sigma_{B_2} \otimes \dots \otimes \sigma_{A_n} \sigma_{B_n}, \quad (3.97)$$

we have that

$$F * G(\vec{n}) = f_{A_1 A_2 \dots A_n} g_{B_1 B_2 \dots B_n} \prod_i (n_{A_i} n_{B_i} + K_{A_i B_i}) \quad (3.98)$$

or

$$\begin{aligned}
 F * G(\vec{n}) = & \\
 FG(\vec{n}) + \sum_{m=1}^n \frac{n!}{m!(n-m)!} & f_{A_1 A_2 \dots A_m A_{m+1} \dots A_n} n_{A_{m+1}} n_{A_{m+2}} \dots n_{A_n} \\
 \times K_{A_1 B_1}(\vec{n}) K_{A_2 B_2}(\vec{n}) \dots & K_{A_m B_m}(\vec{n}) g_{B_1 B_2 \dots B_m B_{m+1} \dots B_n} n_{B_{m+1}} \times \\
 n_{B_{m+2}} \dots n_{B_n} . & \quad (3.99)
 \end{aligned}$$

Now as f and g are symmetric in indices, there is the expression

$$\begin{aligned}
 \partial_{A_1} \partial_{A_2} \dots \partial_{A_m} F(\vec{n}) &= \frac{n!}{(n-m)!} f_{A_1 A_2 \dots A_m A_{m+1} \dots A_n} n_{A_{m+1}} n_{A_{m+2}} \dots n_{A_n} , \\
 \partial_{A_i} &\equiv \frac{\partial}{\partial n_{A_i}} \quad (3.100)
 \end{aligned}$$

for F and a similar expression for G . Hence

$$\begin{aligned}
 F *_C G(\vec{n}) &= \sum_{m=0}^n \frac{(n-m)!}{m!n!} (\partial_{A_1} \partial_{A_2} \dots \partial_{A_m} F)(\vec{n}) \\
 &\times K_{A_1 B_1}(\vec{n}) K_{A_2 B_2}(\vec{n}) \dots K_{A_m B_m}(\vec{n}) (\partial_{B_1} \partial_{B_2} \dots \partial_{B_m} G)(\vec{n}) \quad (3.101)
 \end{aligned}$$

which is the final answer. Here the $m = 0$ term is to be understood as $FG(\vec{n})$, the pointwise product of F and G evaluated at $\vec{n}$. This formula was first given in [63].

Differentiating on n_A ignoring the constraint $\vec{n} \cdot \vec{n} = 1$ is justified in the final answer (3.101) (although not in (3.100)), since $K_{AB}(\vec{n}) \partial_A (\vec{n} \cdot \vec{n}) = K_{AB}(\vec{n}) \partial_B (\vec{n} \cdot \vec{n}) = 0$. (3.100) being only an intermediate step on the way to (3.101), this sloppiness is immaterial.

For large n , (3.101) is an expansion in powers of $\frac{1}{n}$, the leading term giving the commutative product. Thus the algebra S_F^2 is in some sense a deformation of the commutative algebra of functions $C^\infty(S^2)$. But as the maximum angular momentum in F and G is n , we get only the spherical harmonics $Y_{\ell m}$, $\ell \in \{0, 1, \dots, n\}$ in their expansion. For this reason, F and G span a finite-dimensional subspace of $C^\infty(S^2)$ and S_F^2 is not properly a deformation of the commutative algebra $C^\infty(S^2)$.

3.6.2 The Weyl *-Product $*_W$

The Weyl *-products are characterized by the special identity described before. For this reason they are very convenient for use in loop expansions in quantum field theory (see chapter 4).

A simple way to find $*_W$ is to find it via its connection to $*_C$. For this purpose let us consider

$$Tr(\hat{T}_m^\ell)^\dagger \hat{T}_{m'}^{\ell'} = \frac{n+1}{4\pi} \int d\Omega [T_n(\ell)^{\frac{1}{2}} \bar{Y}_{\ell m}] *_C [T_n(\ell')^{\frac{1}{2}} Y_{\ell' m'}](\vec{x}), \quad (3.102)$$

where

$$\langle z, n | T_m^\ell | z, n \rangle = T_n(\ell)^{\frac{1}{2}} Y_{\ell m}(\hat{n}). \quad (3.103)$$

T_m^ℓ is defined in section 2.5 while we can show (3.103) as follows. The factor $T_n(\ell)^{\frac{1}{2}}$ is independent of m by rotational invariance. It is real as shown by complex conjugating (3.103) and using

$$(T_m^\ell)^\dagger = (-1)^m T_{-m}^\ell, \quad \bar{Y}_{\ell m}(\hat{n}) = (-1)^m Y_{\ell, -m}(\hat{n}). \quad (3.104)$$

It can be chosen to be positive as well by explicit calculation. We shall evaluate it later.

The normalization of T_m^ℓ and $Y_{\ell m}$ are

$$Tr(T_m^\ell)^\dagger T_{m'}^{\ell'} = \int d\Omega \bar{Y}_{\ell m}(\vec{x}) Y_{\ell' m'}(\vec{x}) = \delta_{\ell\ell'} \delta_{mm'}. \quad (3.105)$$

Hence using (3.102)

$$\begin{aligned} \delta_{\ell\ell'} \delta_{mm'} &= \int d\Omega \bar{Y}_{\ell m}(\vec{x}) Y_{\ell' m'}(\vec{x}) \\ &= \frac{n+1}{4\pi} \int d\Omega (T_n(\ell)^{\frac{1}{2}} \bar{Y}_{\ell m})(\vec{x}) *_C (T_n(\ell')^{\frac{1}{2}} Y_{\ell' m'})(\vec{x}) \end{aligned} \quad (3.106)$$

Equation (3.106) suggests that the fuzzy sphere algebra $(S_F^2, *_W)$ with the Weyl-Moyal product $*_M$ is obtained from the fuzzy sphere algebra $(S_F^2, *_C)$ with the coherent state $*_C$ product from the map

$$\begin{aligned} \chi : (S_F^2, *_C) &\longrightarrow (S_F^2, *_W) \\ \chi \left(\sqrt{\frac{n+1}{4\pi}} T_n(\ell)^{\frac{1}{2}} Y_{\ell m} \right) &= Y_{\ell m}. \end{aligned} \quad (3.107)$$

The induced $*$, call it for a moment as $*'$, on the image of χ is

$$Y_{\ell m} *_' Y_{\ell' m'} = \chi \left(\sqrt{\frac{n+1}{4\pi}} T_n(\ell)^{\frac{1}{2}} Y_{\ell m} *_C \sqrt{\frac{n+1}{4\pi}} T_n(\ell')^{\frac{1}{2}} Y_{\ell' m'} \right). \quad (3.108)$$

For the evaluation of (3.108), $Y_{\ell m} *_C Y_{\ell' m'}$ has to be written as a series in $Y_{\ell'' m''}$ and χ applied to it term-by-term. We will not need its full details here.

Now replace $Y_{\ell m}$ by $\bar{Y}_{\ell m}$ and integrate. As χ commutes with rotations, only the angular momentum 0 component of $\sqrt{\frac{n+1}{4\pi}} T_n(\ell)^{\frac{1}{2}} \bar{Y}_{\ell m} *_C$

$\sqrt{\frac{n+1}{4\pi}} T_n(\ell')^{\frac{1}{2}} Y_{\ell'm'}$ contributes to the integral. This component is $\delta_{\ell\ell'} \delta_{mm'} \bar{Y}_{00} *_C Y_{00} = \delta_{\ell\ell'} \delta_{mm'} \frac{1}{4\pi}$. Using (3.107) for $\ell = 0$ and the value $T_n(0)^{\frac{1}{2}} = \sqrt{\frac{4\pi}{n+1}}$ to be derived below, we get

$$\int d\Omega \bar{Y}_{\ell m} *_' Y_{\ell' m'} = \delta_{\ell\ell'} \delta_{mm'} = \int d\Omega \bar{Y}_{\ell m} Y_{\ell' m'}. \quad (3.109)$$

Hence $*'$ enjoys the special identity characterizing the Weyl-Moyal product for the basis of functions in our algebra and hence for all functions. $*'$ is the Weyl-Moyal product $*_W$.

T_n is a function $\mathcal{T}_n$ of $\ell(\ell+1)$. The latter is the eigenvalue of $\mathcal{L}^2$, the square of angular momentum. The map χ can hence be defined directly on all functions α by

$$\chi(\alpha) = \sqrt{\frac{n+1}{4\pi}} \mathcal{T}_n(\mathcal{L}^2)^{\frac{1}{2}} \alpha \quad (3.110)$$

where R.H.S. can be calculated for example by expanding α in spherical harmonics.

The evaluation of $T_n^{\frac{1}{2}}(\ell)$ can be done as follows. It is enough to compare the two sides of (3.103) for $m = \ell$. For $m = \ell$,

$$Y_{\ell\ell}(\vec{x}) = \frac{\sqrt{(2\ell+1)!}}{\ell!} \bar{z}_2^\ell z_1^\ell. \quad (3.111)$$

The operator T_ℓ^ℓ being the highest weight state commutes with $L_+ = a_2^\dagger a_1$ while $[L_3, T_\ell^\ell] = \ell T_\ell^\ell$. Hence in terms of a_i and $a_j^\dagger$,

$$T_\ell^\ell = N_\ell a_2^\dagger a_1^\ell \quad (3.112)$$

where the constant N_ℓ is to be fixed by the condition

$$Tr(T_\ell^\ell)^\dagger T_\ell^\ell = 1. \quad (3.113)$$

Evaluating L.H.S. in the basis $\frac{(a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2}}{\sqrt{n_1! n_2!}} |0\rangle$, $n_1 + n_2 = n$, we get after a choice of sign,

$$N_\ell = \sqrt{\frac{4\pi}{n+1}} \frac{(n-\ell)!(n+1)! \sqrt{(2\ell+1)!}}{n!\ell!(n+\ell+1)!} \quad (3.114)$$

and

$$T_\ell^\ell = \sqrt{\frac{4\pi}{n+1}} \frac{(n-\ell)!(n+1)! \sqrt{(2\ell+1)!}}{n!\ell!(n+\ell+1)!} a_2^\dagger a_1^\ell. \quad (3.115)$$

Inserting (3.115) in (3.103) and using (3.111), we get, after a short calculation,

$$T_n(\ell)^{\frac{1}{2}} = \sqrt{\frac{4\pi}{n+1}} \frac{n!(n+1)!}{(n-\ell)!(n+\ell+1)!} \quad (3.116)$$

which gives $T_n(0)^{\frac{1}{2}} = \sqrt{\frac{4\pi}{n+1}}$ as claimed earlier.

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Chapter 4

Scalar Fields on the Fuzzy Sphere

The free Euclidean action for the fuzzy sphere for a scalar field is

$$S_0 = \frac{1}{n+1} \text{Tr} \left[-\frac{1}{2} [L_i, \hat{\phi}] [L_i, \hat{\phi}] + \frac{\mu^2}{2} \hat{\phi}^2 \right] \quad (4.1)$$

where we now have all operators or $(n+1) \times (n+1)$ matrices.

As we saw in chapter 2, the scalar field can be expanded in terms of the polarization tensors $\hat{T}_m^\ell$,

$$\hat{\phi} = \sum_{\ell, m} \phi_{\ell m} \hat{T}_m^\ell, \quad (4.2)$$

where $\phi_{\ell m}$ are complex numbers. For concreteness, we will restrict our attention to Hermitian scalar fields $\hat{\phi}^\dagger = \hat{\phi}$. Since $(\hat{T}_m^\ell)^\dagger = (-1)^m \hat{T}_{-m}^\ell$, this implies that $\bar{\phi}_{\ell, m} = (-1)^m \phi_{\ell, -m}$.

In terms of $\phi_{\ell m}$'s, the action (4.1) is

$$\begin{aligned} S_0 &= \sum_{\ell, m}^{n+1} \frac{|\phi_{\ell m}|^2}{2} (\ell(\ell+1) + \mu^2) \\ &= \sum_{\ell=0}^{n+1} \frac{\phi_{\ell, 0}^2}{2} (\ell(\ell+1) + \mu^2) + 2 \sum_{\ell=0}^{n+1} \sum_{m=1}^{\ell} \frac{|\phi_{\ell m}|^2}{2} (\ell(\ell+1) + \mu^2). \end{aligned} \quad (4.3)$$

The generating function for correlators in this model is

$$Z_0(\hat{J}) = \mathcal{N}_0 \int D\hat{\phi} e^{-S_0 + \frac{1}{n+1} \text{Tr} \hat{J} \hat{\phi}} \quad (4.4)$$

where $\hat{J}$, the “external current” is an $(n+1) \times (n+1)$ Hermitian matrix. Also

$$\mathcal{N}_0 = \left[\int D\hat{\phi} e^{-S_0} \right]^{-1} \quad (4.5)$$