

Lectures on Gravitation

Ashok Das

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Preface

Over the past several years, I have taught a one-semester graduate course on gravitation at the University of Rochester. The principles of general relativity were developed systematically in a traditional manner, and this book comprises my lectures of this course. The class was attended by both graduate as well as several undergraduate students. Correspondingly, the subject matter was developed in detail using tensor analysis (rather than the compact differential geometric concepts). The concepts and ideas of the material were supplemented by a lot of worked out examples dispersed through out the lectures, which the students found extremely useful.

Although it is the normal convention in a course on gravity to use a metric of signature $(-, +, +, +)$, I have used through out the lectures the conventional Bjorken-Drell metric signature $(+, -, -, -)$ for consistency with my other books. Since the book consists of my actual lectures in the course, it is informal with step by step derivation of every result which is quite helpful to students, particularly to undergraduate students.

There are many references to the material covered in this course and it is impractical to give all the references. Instead, let me note only the few following books on the subject which can be used as references and which contain further references on the subject.

1. *Gravitation and Cosmology*, S. Weinberg, John Wiley (1972), New York.
2. *Gravitation*, C. W. Misner, K. S. Thorne and J. A. Wheeler, W. H. Freeman (1973), San Francisco.
3. *Introduction to General Relativity*, R. Adler, M. Bazin and M. Schiffer, McGraw-Hill (1975), New York.

The figures in this book were drawn using Jaxodraw as well as PSTricks. I am grateful to the people who developed these extremely



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useful software. Finally, I would like to thank Fernando Méndez for his help with some of the figures and Dave Munson for sorting out various computer related problems.

Ashok Das
Rochester

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Basics of geometry and relativity

1.1 Two dimensional geometry

We have all heard of the statement that “Gravitation is the study of geometry”. Let us see what this exactly means. We are, of course, familiar with Euclidean geometry. For simplicity we will restrict here only to two dimensions so that the study of geometry of the manifold corresponds to the study of surfaces and Euclidean geometry corresponds simply to the all familiar plane geometry.

A geometry, of course, has built in assumptions. For example, one has to define the basic elements (concepts) of geometry such as points. Furthermore, we know that given any two points we can draw many curves joining them. However, the straight line, as we know, defines the shortest path between two points in Euclidean geometry. Euclid’s axioms, among other things, state that a straight line is determined by two points, it may have indefinite length, that any two right angles are equal, that only one line parallel to a given line can pass through a point outside the line in the same plane, that when equals (equal line segments) are added to equals, the sums are always equal and that the whole is greater than any of its parts. From these axioms Euclid deduced hundreds of theorems which tell us a lot about Euclidean geometry.

For example, let us consider a triangle ABC in plane geometry shown in Fig. 1.1. Through the vertex A let us draw a straight line DAE parallel to BC . From the properties of the parallel lines we immediately conclude that ($\widehat{DAB} = \text{Angle } DAB$)

$$\text{Angle } DAB = \text{Angle } ABC, \quad \text{or,} \quad \widehat{DAB} = \widehat{ABC}, \quad (1.1)$$

and

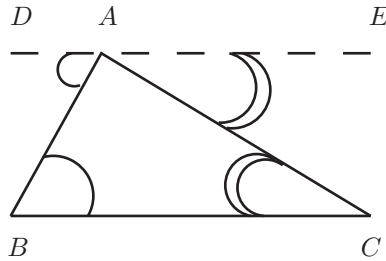


Figure 1.1: A triangle in plane geometry.

$$\text{Angle } EAC = \text{Angle } ACB, \quad \text{or,} \quad \widehat{EAC} = \widehat{ACB}. \quad (1.2)$$

On the other hand, we also know that

$$\widehat{DAB} + \widehat{BAC} + \widehat{EAC} = 180^\circ. \quad (1.3)$$

Thus, we conclude that

$$\widehat{DAB} + \widehat{BAC} + \widehat{EAC} = \widehat{ABC} + \widehat{BAC} + \widehat{ACB} = 180^\circ. \quad (1.4)$$

This, of course, proves the familiar result that the sum of the angles of a triangle on a plane equals 180° . Similarly many more interesting results can be obtained by using Euclid's axioms.

However, it is Euclid's axiom on the parallel lines that generated much interest. It says, "If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines produced indefinitely meet on that side on which the angles are less than two right angles", which is shown in Fig. 1.2. Since this axiom involves extending straight lines to the inexorable region of spatial infinity, people tried hard to prove that this follows from the other axioms of Euclid and that this is not truly an independent axiom.

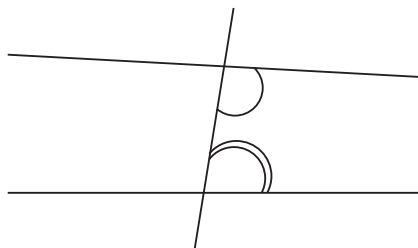


Figure 1.2: Two straight lines in plane geometry which are not parallel according to Euclid's axiom.

The study of this question, on the other hand, led to the birth of other, non-Euclidean geometries which satisfy all the axioms of Euclid except the one on parallel lines. These are geometries of curved surfaces where the definition of a straight line carries over to what we know as geodesics. (As we will see later, geodesics are paths along which the length is extremal.) One such geometry is the geometry of the surface of a sphere. On a sphere the geodesics ("straight lines") are parts of great circles joining the opposite pole points. It is clear that since any two great circles must intersect (at the poles), on a sphere we cannot draw any line parallel to a given line through a point outside of the line.

Another geometry which was developed independently by Gauss, Bolyai and Lobachevski is opposite in its properties to the previous one (namely, the geometry of a sphere). Therefore, it is also known as the geometry of the surface of a pseudosphere. It is rather hard to visualize. But a crude example of this geometry is the surface of a horse saddle. On such a surface, many geodesics can be drawn through a given point not intersecting a given line outside of this point and hence in this geometry, it is possible to draw many lines parallel to a given line through a point outside of the line. These are the three possible distinct geometries we can have.

In a geometry described by curved surfaces it is no longer true that the sum of all angles in a triangle equals 180° . In fact, one can simply show that for a triangle on a sphere the sum of angles is greater than 180° whereas for the geometry of Gauss, Bolyai and Lobachevski, it is less than 180° . Construction of these geometries

showed that the fifth postulate of Euclid is indeed independent.

A question we can ask at this point is what is the geometry of the space we live in. Is it flat, or is it like a sphere or is it like a pseudosphere? If we were two dimensional creatures living in a two dimensional space, we can simply study the properties of geometrical figures and deduce the nature of the geometry ourselves. But this does not seem right. There ought to exist a general quantity which distinguishes one geometry from the others. It must be an inner property of the surface (or the manifold) and let us determine this.

First of all let us note that although a straight line is defined as the shortest path between two points, we still have not defined the concept of a distance. In plane geometry if two points have coordinates (x_1, x_2) and (y_1, y_2) , then we know that the shortest distance between the two is given by

$$d(x, y) = \left[(x_1 - y_1)^2 + (x_2 - y_2)^2 \right]^{\frac{1}{2}}. \quad (1.5)$$

Furthermore, it was shown that for the geometry of Gauss, Bolyai and Lobachevski, a point with coordinates (x_1, x_2) satisfies (suitably scaled so that the coordinates are dimensionless)

$$x_1^2 + x_2^2 < 1, \quad (1.6)$$

and the shortest distance between two points (x_1, x_2) and (y_1, y_2) is given by

$$\cosh \frac{d(x, y)}{a} = \frac{1 - x_1 y_1 - x_2 y_2}{(1 - x_1^2 - x_2^2)^{\frac{1}{2}} (1 - y_1^2 - y_2^2)^{\frac{1}{2}}}. \quad (1.7)$$

Here “ a ” is a length scale set by the problem under study.

Gauss realized that the distance $d(x, y)$ which gives the shortest path between two points determines the inner properties of a space. Thus for example, a cylinder or a cone has the same inner properties as a plane since a plane can be rolled into a cylinder or a cone without stretching or tearing. (The formula for the distance on a cylinder or a cone is the same as on a plane.) However, as we all know, a sphere cannot be unrolled into a plane without tearing

it and correspondingly the formula for the distance on a sphere is different.

Gauss also recognized that any space can be locally identified with a Euclidean space. Hence locally we can set up a Euclidean coordinate system $\xi_i, i = 1, 2$, to describe the space so that the infinitesimal distance between the points (ξ_1, ξ_2) and $(\xi_1 + d\xi_1, \xi_2 + d\xi_2)$ can be written as

$$ds^2 = d\xi_1^2 + d\xi_2^2. \quad (1.8)$$

Of course, if the surface is curved, this coordinate system cannot cover any finite space still satisfying the laws of plane geometry. Let us suppose that there exists a coordinate system (x_1, x_2) which covers the entire surface. Then we can write the distance between two points with the coordinates (x_1, x_2) and $(x_1 + dx_1, x_2 + dx_2)$ (which locally coincides with (ξ_1, ξ_2) and $(\xi_1 + d\xi_1, \xi_2 + d\xi_2)$) as

$$\begin{aligned} ds^2 &= d\xi_1^2 + d\xi_2^2 \\ &= \left(\frac{\partial \xi_1}{\partial x_1} dx_1 + \frac{\partial \xi_1}{\partial x_2} dx_2 \right)^2 + \left(\frac{\partial \xi_2}{\partial x_1} dx_1 + \frac{\partial \xi_2}{\partial x_2} dx_2 \right)^2 \\ &= \left[\left(\frac{\partial \xi_1}{\partial x_1} \right)^2 + \left(\frac{\partial \xi_2}{\partial x_1} \right)^2 \right] dx_1^2 \\ &\quad + 2 \left[\frac{\partial \xi_1}{\partial x_1} \frac{\partial \xi_1}{\partial x_2} + \frac{\partial \xi_2}{\partial x_1} \frac{\partial \xi_2}{\partial x_2} \right] dx_1 dx_2 \\ &\quad + \left[\left(\frac{\partial \xi_1}{\partial x_2} \right)^2 + \left(\frac{\partial \xi_2}{\partial x_2} \right)^2 \right] dx_2^2 \\ &= g_{11}(x) dx_1^2 + 2g_{12}(x) dx_1 dx_2 + g_{22}(x) dx_2^2 \\ &= g_{ij} dx_i dx_j, \end{aligned} \quad (1.9)$$

where we have identified

$$\begin{aligned}
g_{11}(x) &= \left(\frac{\partial \xi_1}{\partial x_1} \right)^2 + \left(\frac{\partial \xi_2}{\partial x_1} \right)^2, \\
g_{12}(x) &= \frac{\partial \xi_1}{\partial x_1} \frac{\partial \xi_1}{\partial x_2} + \frac{\partial \xi_2}{\partial x_1} \frac{\partial \xi_2}{\partial x_2} = g_{21}(x), \\
g_{22}(x) &= \left(\frac{\partial \xi_1}{\partial x_2} \right)^2 + \left(\frac{\partial \xi_2}{\partial x_2} \right)^2.
\end{aligned} \tag{1.10}$$

The form of ds^2 in (1.9) is the sign of a metric space and $g_{ij}, i, j = 1, 2$ represents the metric tensor of the two dimensional space. Thus, for example, the metric for a plane where

$$ds^2 = dx_1^2 + dx_2^2, \tag{1.11}$$

is given by

$$g_{11} = g_{22} = 1, \quad g_{12} = 0. \tag{1.12}$$

For a sphere of fixed radius r , on the other hand,

$$ds^2 = r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \tag{1.13}$$

Thus, in this case, the metric takes the form

$$g_{\theta\theta} = r^2, \quad g_{\theta\phi} = 0, \quad g_{\phi\phi} = r^2 \sin^2 \theta. \tag{1.14}$$

It is this dependence of the metric on the coordinate θ that gives the sphere different inner properties from that of a plane. We can also show that for the geometry of Gauss, Bolyai and Lobachevski, the metric is given by

$$\begin{aligned}
g_{11}(x) &= \frac{a^2 (1 - x_2^2)}{(1 - x_1^2 - x_2^2)^2}, \\
g_{12}(x) &= \frac{a^2 x_1 x_2}{(1 - x_1^2 - x_2^2)^2}, \\
g_{22}(x) &= \frac{a^2 (1 - x_1^2)}{(1 - x_1^2 - x_2^2)^2}.
\end{aligned} \tag{1.15}$$

Once we know the metric functions, the length of a path can be obtained by integrating ds along that path. Thus the metric functions determine all the inner properties of a space. Unfortunately, however, they are also coordinate dependent. Note that if we parameterize a plane with Cartesian coordinates as in (1.12), then we have

$$g_{11} = g_{22} = 1, \quad \text{and} \quad g_{12} = 0. \quad (1.16)$$

However, if we parameterize the same surface with polar coordinates, then we can write

$$ds^2 = dr^2 + r^2 d\theta^2, \quad (1.17)$$

so that

$$g_{rr} = 1, \quad g_{\theta\theta} = r^2, \quad \text{and} \quad g_{r\theta} = 0. \quad (1.18)$$

This does not look like an Euclidean space (the metric component $g_{\theta\theta}$ is a function of coordinates), but it really is the same space. This example shows that although the metric functions determine all the inner properties of a space, since their forms depend on the particular coordinate system used, they are not very useful in classifying spaces. We should look for an object constructed from g_{ij} that does not depend on the choice of coordinate system.

In fact, Gauss had found a unique function (in two dimensions) that does this. It is called the Gaussian curvature and is defined (for two dimensional spaces) as

$$\begin{aligned} K(x_1, x_2) = & \frac{1}{2g} \left[2 \frac{\partial^2 g_{12}}{\partial x_1 \partial x_2} - \frac{\partial^2 g_{11}}{\partial x_2^2} - \frac{\partial^2 g_{22}}{\partial x_1^2} \right] \\ & - \frac{g_{11}}{4g^2} \left[\left(\frac{\partial g_{22}}{\partial x_2} \right) \left(2 \frac{\partial g_{12}}{\partial x_1} - \frac{\partial g_{11}}{\partial x_2} \right) - \left(\frac{\partial g_{22}}{\partial x_1} \right)^2 \right] \\ & - \frac{g_{22}}{4g^2} \left[\left(\frac{\partial g_{11}}{\partial x_1} \right) \left(2 \frac{\partial g_{12}}{\partial x_2} - \frac{\partial g_{22}}{\partial x_1} \right) - \left(\frac{\partial g_{11}}{\partial x_2} \right)^2 \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{g_{12}}{4g^2} \left[\left(\frac{\partial g_{11}}{\partial x_1} \right) \left(\frac{\partial g_{22}}{\partial x_2} \right) - 2 \left(\frac{\partial g_{11}}{\partial x_2} \right) \left(\frac{\partial g_{22}}{\partial x_1} \right) \right. \\
& \quad \left. + \left(2 \frac{\partial g_{12}}{\partial x_1} - \frac{\partial g_{11}}{\partial x_2} \right) \left(2 \frac{\partial g_{12}}{\partial x_2} - \frac{\partial g_{22}}{\partial x_1} \right) \right].
\end{aligned} \tag{1.19}$$

Here g represents the determinant of the two dimensional metric g_{ij} , namely,

$$g = \det g_{ij} = g_{11}g_{22} - g_{12}^2. \tag{1.20}$$

It is clear from the expressions in (1.19) as well as (1.16) that the Gaussian curvature vanishes for a plane surface (geometry),

$$K = 0. \tag{1.21}$$

For a sphere, on the other hand, we note from (1.14) that (we can identify $x_1 = \theta, x_2 = \phi, g_{11} = g_{\theta\theta}, g_{22} = g_{\phi\phi}, g_{12} = g_{\theta\phi}$)

$$\begin{aligned}
g &= r^4 \sin^2 \theta, \\
\frac{\partial g_{\theta\theta}}{\partial \theta} &= \frac{\partial g_{\theta\theta}}{\partial \phi} = \frac{\partial g_{\phi\phi}}{\partial \phi} = 0, \\
\frac{\partial g_{\phi\phi}}{\partial \theta} &= 2r^2 \sin \theta \cos \theta = r^2 \sin 2\theta, \\
\frac{\partial^2 g_{\phi\phi}}{\partial \theta^2} &= 2r^2 \cos 2\theta.
\end{aligned} \tag{1.22}$$

Thus, for a sphere we obtain

$$\begin{aligned}
K &= \frac{1}{2g} \left(-\frac{\partial^2 g_{\phi\phi}}{\partial \theta^2} \right) - \frac{g_{\theta\theta}}{4g^2} \left(-\left(\frac{\partial g_{\phi\phi}}{\partial \theta} \right)^2 \right) \\
&= \frac{1}{2r^4 \sin^2 \theta} (-2r^2 \cos 2\theta) - \frac{r^2}{4r^8 \sin^4 \theta} (-r^4 \sin^2 2\theta) \\
&= -\frac{1}{r^2 \sin^2 \theta} (1 - 2 \sin^2 \theta) + \frac{1}{4r^2 \sin^4 \theta} (4 \sin^2 \theta \cos^2 \theta)
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{r^2 \sin^2 \theta} (1 - 2 \sin^2 \theta - \cos^2 \theta) \\
&= -\frac{1}{r^2 \sin^2 \theta} (-\sin^2 \theta) = \frac{1}{r^2} > 0.
\end{aligned} \tag{1.23}$$

That is, the surface of a sphere is a space of constant (since r is a fixed constant) positive curvature. Similarly, one can show that the surface of a pseudosphere has a constant negative curvature,

$$K = -\frac{1}{a^2} < 0. \tag{1.24}$$

We can check that the Gaussian curvature is independent of the coordinate system used by simply calculating the curvature of a plane in polar coordinates. Let us note from (1.18) that, in this case, (we can identify $x_1 = r, x_2 = \theta, g_{11} = g_{rr}, g_{22} = g_{\theta\theta}, g_{12} = g_{r\theta}$)

$$\begin{aligned}
g &= r^2, \\
\frac{\partial g_{rr}}{\partial r} &= \frac{\partial g_{rr}}{\partial \theta} = \frac{\partial g_{\theta\theta}}{\partial \theta} = 0, \\
\frac{\partial g_{\theta\theta}}{\partial r} &= 2r, \\
\frac{\partial^2 g_{\theta\theta}}{\partial r^2} &= 2.
\end{aligned} \tag{1.25}$$

Thus, in these coordinates

$$\begin{aligned}
K(r, \theta) &= \frac{1}{2g} \left[-\frac{\partial^2 g_{\theta\theta}}{\partial r^2} \right] - \frac{g_{rr}}{4g^2} \left[-\left(\frac{\partial g_{\theta\theta}}{\partial r} \right)^2 \right] \\
&= \frac{1}{2r^2} [-2] - \frac{1}{4r^4} [-(2r)^2] \\
&= -\frac{1}{r^2} + \frac{1}{r^2} = 0.
\end{aligned} \tag{1.26}$$

Thus we see that if we can calculate the Gaussian curvature of a surface, we can distinguish between the three geometries independent of the choice of coordinate system.

So far we have been dealing with only two dimensions and hence there is only one quantity independent of the coordinate system used. This can be intuitively seen from the fact that since the metric is symmetric, in two dimensions, there can only be three independent metric components. There are, however, two coordinates to represent any system. Hence the number of quantities that can be formed which are independent of the coordinates used is exactly one (namely, $3 - 2 = 1$). In higher dimensions, say D dimensions, the number of independent metric components is

$$\frac{1}{2} D(D + 1). \quad (1.27)$$

However, since there are D coordinates to specify a system, the number of independent quantities that can be constructed which are independent of the coordinate system used is

$$\frac{1}{2} D(D + 1) - D = \frac{1}{2} D(D - 1). \quad (1.28)$$

As a result, the study of geometry becomes more complicated when $D > 2$. This general problem was solved by Riemann and was extended to physics by Einstein.

► **Example (Pseudosphere in two dimensions).** For the pseudosphere in two dimensions we have (see (1.7))

$$\cosh \frac{d(x, y)}{a} = \frac{1 - x_1 y_1 - x_2 y_2}{(1 - x_1^2 - x_2^2)^{\frac{1}{2}} (1 - y_1^2 - y_2^2)^{\frac{1}{2}}}, \quad (1.29)$$

where $d(x, y)$ denotes the distance between the points (x_1, x_2) and (y_1, y_2) . We would like to determine the metric as well as the Gaussian curvature for this space.

For any two vectors A, B in two dimensions, let us introduce the compact notation

$$A \cdot B = A_i B_i = A_1 B_1 + A_2 B_2, \quad A^2 = A_i A_i = A_1 A_1 + A_2 A_2, \quad (1.30)$$

where $i = 1, 2$ and we use the summation convention over repeated indices. Then we can write (1.29) as

$$\cosh \frac{d(x, y)}{a} = \frac{1 - x \cdot y}{\sqrt{1 - x^2} \sqrt{1 - y^2}}. \quad (1.31)$$

To determine the metric tensor for this space, we need to consider only infinitesimal distances (see discussion following (1.7)) so that let us assume $y_i = x_i + dx_i$, $i = 1, 2$ and we can identify the infinitesimal distance with the proper length $d(x, y) = ds$. Expanding both sides of the equation (1.31) and keeping terms to quadratic order in the infinitesimal changes, we obtain

$$\begin{aligned}
 \cosh \frac{ds}{a} &= 1 + \frac{ds^2}{2a^2} + \dots \\
 &= \frac{(1 - x^2 - x \cdot dx)}{1 - x^2} \left[1 + \frac{1}{2} \frac{2x \cdot dx + dx^2}{1 - x^2} + \frac{3}{8} \left(\frac{2x \cdot dx + dx^2}{1 - x^2} \right)^2 + \dots \right] \\
 &= \left(1 - \frac{x \cdot dx}{1 - x^2} \right) \left[1 + \frac{1}{2} \frac{2x \cdot dx + dx^2}{1 - x^2} + \frac{3}{8} \left(\frac{2x \cdot dx + dx^2}{1 - x^2} \right)^2 + \dots \right] \\
 &= 1 + \frac{1}{2} \frac{dx^2}{1 - x^2} + \frac{1}{2} \frac{(x \cdot dx)^2}{(1 - x^2)^2}, \tag{1.32}
 \end{aligned}$$

where $dx^2 = dx \cdot dx$. Note that the linear terms (in the differentials) on the right hand side of (1.32) in the expansion for the distance cancel and the quadratic terms lead to

$$\begin{aligned}
 \frac{ds^2}{2a^2} &= \frac{1}{2} \frac{dx^2}{1 - x^2} + \frac{1}{2} \frac{(x \cdot dx)^2}{(1 - x^2)^2} \\
 &= \frac{1}{2(1 - x^2)} \left(\delta_{ij} + \frac{x_i x_j}{(1 - x^2)} \right) dx_i dx_j, \tag{1.33}
 \end{aligned}$$

where $i, j = 1, 2$ and summation over repeated indices is understood. This is known as the line element for the space and determines the metric components for the pseudosphere to be

$$g_{11} = \frac{a^2(1 - x_2^2)}{(1 - x^2)^2}, \quad g_{12} = g_{21} = \frac{x_1 x_2 a^2}{(1 - x^2)^2}, \quad g_{22} = \frac{a^2(1 - x_1^2)}{(1 - x^2)^2}. \tag{1.34}$$

Starting with the components of the metric tensor in (1.34), we obtain the determinant of the metric to be (see (1.20))

$$\begin{aligned}
 g &= g_{11}g_{22} - g_{12}^2 \\
 &= \frac{a^4}{(1 - x^2)^4} [(1 - x_1^2)(1 - x_2^2) - x_1^2 x_2^2] \\
 &= \frac{a^4}{(1 - x^2)^3}. \tag{1.35}
 \end{aligned}$$

Next let us evaluate the three expressions involved in the calculation of the Gaussian curvature (see (1.19)). For simplicity, let us define A , B_{12} , and C as

$$\begin{aligned}
A &= 2 \frac{\partial^2 g_{12}}{\partial x_1 \partial x_2} - \frac{\partial^2 g_{11}}{\partial x_2^2} - \frac{\partial^2 g_{22}}{\partial x_1^2} \\
&= \frac{2a^2}{(1-x^2)^2} + \frac{8a^2 x^2}{(1-x^2)^3} + \frac{48a^2 x_1^2 x_2^2}{(1-x^2)^4} \\
&\quad + \frac{2a^2}{(1-x^2)^2} - \frac{4a^2(1-5x_2^2)}{(1-x^2)^3} - \frac{24a^2 x_2^2(1-x_2^2)}{(1-x^2)^4} \\
&\quad + \frac{2a^2}{(1-x^2)^2} - \frac{4a^2(1-5x_1^2)}{(1-x^2)^3} - \frac{24a^2 x_1^2(1-x_1^2)}{(1-x^2)^4} \\
&= \frac{2a^2}{(1-x^2)^2} - \frac{4a^2}{(1-x^2)^3} = -\frac{2a^2(1+x^2)}{(1-x^2)^3}, \tag{1.36}
\end{aligned}$$

$$\begin{aligned}
B_{12} &= \frac{\partial g_{22}}{\partial x_2} \left(2 \frac{\partial g_{12}}{\partial x_1} - \frac{\partial g_{11}}{\partial x_2} \right) - \left(\frac{\partial g_{22}}{\partial x_1} \right)^2 \\
&= \frac{4a^2 x_2(1-x_1^2)}{(1-x^2)^3} \left[\frac{2a^2 x_2}{(1-x^2)^2} + \frac{8a^2 x_1^2 x_2}{(1-x^2)^3} + \frac{2a^2 x_2}{(1-x^2)^2} \right. \\
&\quad \left. - \frac{4a^2 x_2(1-x_2^2)}{(1-x^2)^3} \right] - \left[-\frac{2a^2 x_1}{(1-x^2)^2} + \frac{4a^2 x_1(1-x_1^2)}{(1-x^2)^3} \right]^2 \\
&= -\frac{4a^4 x_1^2}{(1-x^2)^4}, \tag{1.37}
\end{aligned}$$

$$\begin{aligned}
C &= \frac{\partial g_{11}}{\partial x_1} \frac{\partial g_{22}}{\partial x_2} - 2 \frac{\partial g_{11}}{\partial x_2} \frac{\partial g_{22}}{\partial x_1} + \left(2 \frac{\partial g_{12}}{\partial x_1} - \frac{\partial g_{11}}{\partial x_2} \right) \left(2 \frac{\partial g_{12}}{\partial x_2} - \frac{\partial g_{22}}{\partial x_1} \right) \\
&= \frac{4a^2 x_1(1-x_2^2)}{(1-x^2)^3} \frac{4a^2 x_2(1-x_1^2)}{(1-x^2)^3} \\
&\quad - 2 \left[-\frac{2a^2 x_2}{(1-x^2)^2} + \frac{4a^2 x_2(1-x_2^2)}{(1-x^2)^3} \right] \times (1 \leftrightarrow 2) \\
&\quad + \left[\frac{2a^2 x_2}{(1-x^2)^2} + \frac{8a^2 x_2^2 x_1^2}{(1-x^2)^3} + \frac{2a^2 x_2}{(1-x^2)^2} - \frac{4a^2 x_2(1-x_2^2)}{(1-x^2)^3} \right] \times (1 \leftrightarrow 2) \\
&= \frac{8a^4 x_1 x_2}{(1-x^2)^4}. \tag{1.38}
\end{aligned}$$

Substituting (1.36)-(1.38) into (1.19), we obtain the Gaussian curvature K to be

$$\begin{aligned}
K &= \frac{1}{2g} A - \frac{g_{11}}{4g^2} B_{12} - \frac{g_{22}}{4g^2} B_{21} + \frac{g_{12}}{4g^2} C \\
&= \frac{1}{2 \frac{a^4}{(1-x^2)^3}} \left[-\frac{2a^2(1+x^2)}{(1-x^2)^3} \right]
\end{aligned}$$

$$\begin{aligned}
& - \frac{1}{4 \left(\frac{a^4}{(1-x^2)^3} \right)^2} \left[- \frac{4a^6 x_1^2 (1-x_2^2)}{(1-x^2)^6} - (1 \leftrightarrow 2) \right] \\
& + \frac{\frac{x_1 x_2 a^2}{(1-x^2)^2}}{4 \left(\frac{a^4}{(1-x^2)^3} \right)^2} \frac{8a^4 x_1 x_2}{(1-x^2)^4} \\
& = - \frac{1}{a^2} (1 + x^2 - x_1^2(1-x_2^2) - x_2^2(1-x_1^2) - 2x_1^2 x_2^2) \\
& = - \frac{1}{a^2} (1 + x^2 - x^2) = - \frac{1}{a^2},
\end{aligned} \tag{1.39}$$

which coincides with (1.24). ◀

1.2 Inertial and gravitational masses

Newton's second law of motion says that any force applied on a particle is proportional to the acceleration it produces. Hence we can write

$$\mathbf{F} = m_I \mathbf{a}. \tag{1.40}$$

This refers to the inertial properties of the system since the force can be any force in general. However, when a particle is subjected to a gravitational pull, it experiences a force given by

$$\mathbf{F} = m_g \mathbf{g}, \tag{1.41}$$

where $\mathbf{g}$ represents the acceleration due to gravity. Here m_I and m_g are constants of proportionality known respectively as the inertial and the gravitational mass of the particle and can, in principle, be different. However, if we combine both the equations (1.40) and (1.41), we deduce that the acceleration produced by a gravitational force has the form

$$\mathbf{a} = \left(\frac{m_g}{m_I} \right) \mathbf{g}. \tag{1.42}$$

If the inertial mass does not coincide with the gravitational mass ($m_I \neq m_g$) and $\frac{m_g}{m_I}$ is not the same for different particles, then

clearly different objects can fall with different accelerations under the influence of gravity, the pendula of the same length but made of different material can have different time periods. On the other hand, if $\frac{m_g}{m_I}$ is the same for all materials, they would fall with the same acceleration. Newton tried to measure this ratio with pendula of different material and Fresnel improved on it. It has been checked by now that (in fact, it holds even to 10^{-13})

$$\left| \frac{m_g}{m_I} - 1 \right| < 10^{-11}, \quad (1.43)$$

for different materials.

Let us examine the experimental set up which measures this ratio. The experiment was designed by Eötvös of Hungary and consists of suspending two weights of different material from a finely balanced beam as shown in Fig. 1.3. Here g' and g'_s represent the vertical and the horizontal components of the centrifugal acceleration due to the rotation of earth as shown in Fig. 1.4 (g'_s is along the surface).

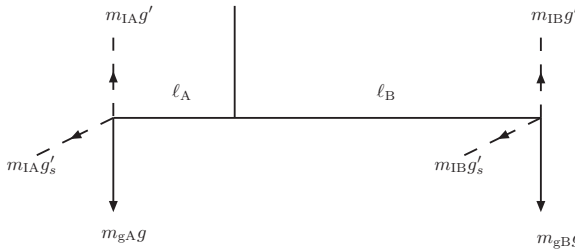


Figure 1.3: Experimental setup for measuring $\frac{m_g}{m_I}$.

The beam is in equilibrium so that we have

$$\ell_A (m_{gA} g - m_{IA} g') = \ell_B (m_{gB} g - m_{IB} g'). \quad (1.44)$$

Therefore, we can solve for ℓ_B from (1.44) to obtain

$$\ell_B = \ell_A \frac{(m_{gA} g - m_{IA} g')}{(m_{gB} g - m_{IB} g')}. \quad (1.45)$$

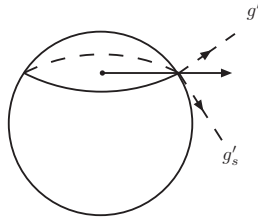


Figure 1.4: Vertical and horizontal components of the centrifugal acceleration.

On the other hand, the horizontal component of the centrifugal acceleration would lead to a torque

$$\begin{aligned}
 T &= \ell_A m_{IA} g'_s - \ell_B m_{IB} g'_s \\
 &= g'_s \left(\ell_A m_{IA} - \ell_A \frac{(m_{gA} g - m_{IA} g')}{m_{gB} g - m_{IB} g'} m_{IB} \right) \\
 &= \ell_A g'_s \left(\frac{m_{IA} m_{gB} - m_{IB} m_{gA}}{m_{gB} g - m_{IB} g'} \right) g,
 \end{aligned} \tag{1.46}$$

where we have used (1.45). Furthermore, if we assume that the vertical component of the centrifugal acceleration is much smaller than the acceleration due to gravity ($g' \ll g$), we can write

$$\begin{aligned}
 T &= \ell_A g'_s \left(m_{IA} - \frac{m_{IB} m_{gA}}{m_{gB}} \right) \\
 &= m_{gA} \ell_A g'_s \left(\frac{m_{IA}}{m_{gA}} - \frac{m_{IB}}{m_{gB}} \right).
 \end{aligned} \tag{1.47}$$

Any inequality in the ratio $\frac{m_I}{m_g}$ for the two weights would lead to a twist in the wire from which the beam is suspended. No twist was observed and Eötvös concluded from this that the difference of $\frac{m_I}{m_g}$ (or conversely $\frac{m_g}{m_I}$) from unity can be at most one part in 10^9 . Dicke and his coworkers later improved the experiment and the limit we presently have is one part in 10^{11} (or even 10^{13}). This is, of course, an impressive result because *a priori* we do not know why

the gravitational mass and the inertial mass should be equal to such a high precision. (We note that if $\frac{m_g}{m_I}$ is a constant for all materials, then this constant can be absorbed into the definition of Newton's constant and we can identify $\frac{m_g}{m_I} = 1$.)

1.3 Relativity

The Aristotelian view, the Galilean view and the modern view of relativity have one point in common. Each assumes that space-time is a collection of events and that each event is labelled by three spatial coordinates and one time coordinate. That is to say, the space-time manifold is a four dimensional continuum. However, beyond this the different views differ drastically from one another.

For example, in the Aristotelian view, events take place at absolute space and time coordinates and three different kinds of events can take place in such a manifold. An Aristotelian observer would observe an event A to precede event B or follow event B or happen simultaneously with event B . All events that happen simultaneously can happen anywhere in space and hence define a three dimensional hypersurface. Therefore, the causal structure of space-time is described as in Fig. 1.5.

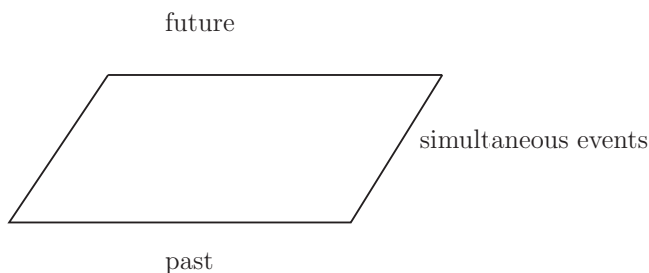


Figure 1.5: Causal structure of space-time in Aristotelian and Galilean relativity.

On the other hand, if we look at Newton's law of motion for a particle under the influence of gravitational forces, the equation has the form

$$m \frac{d^2 \mathbf{x}}{dt^2} = \sum_N \frac{m m_N (\mathbf{x}_N - \mathbf{x})}{|\mathbf{x}_N - \mathbf{x}|^3}, \quad (1.48)$$

where we are assuming that the gravitational force is due to a distribution of point masses in space with $\mathbf{x}_N$ representing the location of the mass m_N . We see that Newton's law selects out a family of inertial frames of reference. For example, let us consider the transformation

$$\begin{aligned} \mathbf{x} &\rightarrow R\mathbf{x} + \mathbf{v}t + \mathbf{a}, & \det R &= 1, \\ t &\rightarrow t + T, \end{aligned} \quad (1.49)$$

where R is a rotation matrix (constant, independent of $\mathbf{x}, t$), $\mathbf{v}$, $\mathbf{a}$ and T are constants. Clearly, then, under such a transformation,

$$\begin{aligned} \frac{d^2 \mathbf{x}}{dt^2} &\rightarrow R \frac{d^2 \mathbf{x}}{dt^2}, \\ \frac{(\mathbf{x}_N - \mathbf{x})}{|\mathbf{x}_N - \mathbf{x}|^3} &\rightarrow \frac{R(\mathbf{x}_N - \mathbf{x})}{|\mathbf{x}_N - \mathbf{x}|^3}. \end{aligned} \quad (1.50)$$

Therefore, Newton's equation (1.48) remains invariant under the transformation (1.49) which is known as the Galilean transformation. It is a ten parameter family (group) of transformations (three parameters of rotation, three velocity parameters $\mathbf{v}$, three parameters of translation $\mathbf{a}$ for the coordinates and one time translation parameter T) and Newton's equations are invariant under Galilean transformations. We note that Newton's equation would not be invariant if we go to a rotating or linearly accelerating frame, i.e., if R and $\mathbf{v}$ are time dependent.

Galilean transformations select out a family of inertial frames in which Newton's equations (laws) are valid. An observer in such a frame would observe an event A to precede an event B or follow B or happen simultaneously with B . In this sense it is the same as the view of the Aristotelian observer and the causal structure of space-time still remains the same as in Fig. 1.5. However, unlike the Aristotelian observer, no two Galilean observers in two different inertial frames

would agree on the exact position or the time of an event. The only thing that they can agree on is the time interval between two events and the space interval between them. Furthermore, we note that in the Galilean view neither the concept of a constant absolute speed nor the concept of a limiting speed exists (namely, frames can move with any arbitrary velocity $\mathbf{v}$).

According to Newton the view of space-time was the following. He believed that there is an absolute fixed space-time and the inertial forces are due to the frames of reference accelerating with respect to the absolute space-time. He tried to justify his view through the example of a rotating bucket of water. A great number of arguments followed and the one that stands out in particular is known as Mach's principle. This says that the acceleration experienced by a particle is influenced by the distribution of matter in space-time. That is, the presence of big stars would change the acceleration experienced by a particle. We can contrast this with the Newtonian view which would say that the presence of big stars does not matter to the acceleration of a particle. We will see later during the course of these lectures how Einstein's theory reconciles both these ideas.

The Newtonian theory along with Galilean relativity, namely that the relative separation of events in space and time are meaningful, met with a large number of success. The place where theoretical prediction disagreed with observation was in the precession of the perihelion of mercury. The theoretical calculations using Newton's equations yielded a value of $8''/\text{century}$ for the value of the precession in contrast to the observed value of $43''/\text{century}$. (We will do this calculation later using Einstein's general relativity.)

We note here only two other phenomena that would not conform to the views of Galilean relativity (the first was observed much later). We experimentally observe that μ mesons decay at rest in the laboratory with a lifetime $\tau_\mu = 10^{-6}$ sec. On the other hand, μ mesons are also produced in cosmic ray showers. And it was measured that the cosmic ray muons have a lifetime about ten times longer than the measured lifetime in the laboratory. Various explanations were put forward to explain this, one of which is that the muon lifetime changes with the speed of the particle. This was indeed verified in the laboratory by producing muons at various speeds. However, this poses a serious problem to Galilean relativity since we assume that observers in different inertial frames have clocks which read the same

time intervals and hence can be compared. The μ meson example, on the other hand, shows that if we regard the muon to be a clock (in the sense that it defines a time interval of 10^{-6} sec) then the time reading depends on whether the muon is at rest or is in motion.

The second problem comes from Maxwell's equations which, in vacuum (without sources), take the form

$$\begin{aligned}\nabla \cdot \mathbf{E} &= 0, \\ \nabla \cdot \mathbf{B} &= 0, \\ \nabla \times \mathbf{E} &= -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \\ \nabla \times \mathbf{B} &= \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}.\end{aligned}\tag{1.51}$$

First of all it is straightforward to check that these equations are no longer invariant under the Galilean transformations (1.49). Secondly, equations (1.51) lead to

$$\begin{aligned}\nabla \times (\nabla \times \mathbf{E}) &= -\frac{1}{c} \frac{\partial}{\partial t} (\nabla \times \mathbf{B}), \\ \text{or, } \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} &= -\frac{1}{c} \frac{\partial}{\partial t} \left(\frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \right), \\ \text{or, } \nabla^2 \mathbf{E} &= \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}.\end{aligned}\tag{1.52}$$

This predicts that electromagnetic waves propagate with the constant speed of light c . A constant absolute speed is, of course, against the spirit of Galilean relativity and hence Maxwell postulated that electromagnetic waves propagate through a medium called ether. That is, one selects out a unique frame of reference to define the speed of light. However, experiments failed to detect ether and the Michelson-Morley experiment was decisive in establishing that there was no ether and that the speed of light nonetheless is an absolute constant.

These conflicts, among other issues, led people to reevaluate the Galilean view and Einstein proposed to replace the requirement of invariance of physical laws under Galilean transformations by that

under Lorentz transformations. This goes under the name of special theory of relativity. The first step in this direction and a great one is to realize that space and time should be treated on equal footing. Thus one defines a four dimensional space-time manifold as well as a coordinate four vector (four dimensional vector) on this manifold of the form

$$x^\mu = (ct, \mathbf{x}), \quad \mu = 0, 1, 2, 3. \quad (1.53)$$

The length of a vector on this manifold is defined as

$$x^2 = \eta_{\mu\nu} x^\mu x^\nu, \quad (1.54)$$

where $\eta_{\mu\nu}$ is the metric of the four dimensional space known as the Minkowski space. The metric is symmetric and diagonal with components

$$\begin{aligned} \eta_{\mu\nu} &= 0, \quad \text{if } \mu \neq \nu, \\ \eta_{00} &= -\eta_{ii} = 1, \quad i = 1, 2, 3, \quad i \text{ not summed.} \end{aligned} \quad (1.55)$$

We can define the inverse metric $\eta^{\mu\nu}$ from the relation

$$\eta^{\mu\nu} \eta_{\nu\gamma} = \delta_\gamma^\mu. \quad (1.56)$$

It is easy to see that the inverse metric also has the form

$$\begin{aligned} \eta^{\mu\nu} &= 0, \quad \text{if } \mu \neq \nu, \\ \eta^{00} &= -\eta^{ii} = 1, \quad i \text{ not summed.} \end{aligned} \quad (1.57)$$

Note that we can raise and lower the index of a given four vector with the help of the metric as

$$\begin{aligned} x_\mu &= \eta_{\mu\nu} x^\nu, \\ x^\mu &= \eta^{\mu\nu} x_\nu, \end{aligned} \quad (1.58)$$

where repeated indices are summed and this leads to

$$x_\mu = (ct, -\mathbf{x}). \quad (1.59)$$

The two four vectors x^μ, x_μ are known respectively as contravariant and covariant four vectors and the length of the coordinate vector (1.54) takes the form

$$x^2 = \eta^{\mu\nu} x_\mu x_\nu = \eta_{\mu\nu} x^\mu x^\nu = x^\mu x_\mu = c^2 t^2 - \mathbf{x}^2. \quad (1.60)$$

Clearly x^2 must be a four dimensional scalar since it does not carry any index. (We note that unlike the length of a vector in the Euclidean space, here the length need not be positive since it involves a difference of two terms.)

We can ask for the form of the four dimensional transformation which leaves this length invariant. It is clear that if

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu, \quad (1.61)$$

then the length remains invariant under this transformation only if

$$\begin{aligned} x'^2 &= \eta_{\mu\nu} x'^\mu x'^\nu = x^2 = \eta_{\mu\nu} x^\mu x^\nu, \\ \text{or, } \eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma x^\rho x^\sigma &= \eta_{\mu\nu} x^\mu x^\nu, \\ \text{or, } \eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma &= \eta_{\rho\sigma}. \end{aligned} \quad (1.62)$$

This has the same form as $\Lambda^T \eta \Lambda = \eta$ in matrix notation (the superscript T denotes matrix transposition) which defines rotations in this manifold. Therefore, we conclude that rotations in the four dimensional space-time manifold which leave the length invariant also leave the metric invariant. These are known as homogeneous Lorentz transformations (they consist of three dimensional rotations and boosts).

However, in addition to four dimensional rotations if we also translate the coordinate axes by a constant amount, we obtain the inhomogeneous Lorentz transformations (also known as Poincaré transformations) under which

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu. \quad (1.63)$$

The infinitesimal length interval between two points (as well as the finite length interval) remains invariant under this transformation, namely,

$$\begin{aligned} ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu &\rightarrow ds'^2 = \eta_{\mu\nu} dx'^\mu dx'^\nu \\ &= \eta_{\mu\nu} \Lambda^\mu_\rho dx^\rho \Lambda^\nu_\sigma dx^\sigma \\ &= \eta_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma dx^\rho dx^\sigma \\ &= \eta_{\rho\sigma} dx^\rho dx^\sigma = ds^2 = c^2 dt^2 - d\mathbf{x}^2. \end{aligned} \quad (1.64)$$

Here ds is known as the “proper length” of the interval. As we know, Lorentz transformations mix up space and time coordinates and hence neither a space interval nor a time interval remains invariant under these transformations. However, it is the particular combination of the space and time interval given by the proper length in (1.64) which is invariant under an inhomogeneous Lorentz transformation (Poincaré transformation). (We can contrast this with the Galilean view according to which different observers will agree on both space and time intervals.)

For a light wave, the wave velocity is given by $|\frac{d\mathbf{x}}{dt}| = c$ and hence the proper length for the path of a light wave (photon) follows from (1.64) to vanish

$$ds = 0. \quad (1.65)$$

If we go to a different inertial frame the proper length does not change so that for a light wave we still have

$$ds' = 0. \quad (1.66)$$

Hence in a different Lorentz frame the speed of light would still be equal to c . That is, Lorentz transformations can accommodate a constant absolute speed and the speed of light can be the same irrespective of the inertial frame of reference we choose. This is of



course, the content of the Michelson-Morley experiment. Clearly since this represents a departure from the Galilean view, the causal structure of space-time must be different. Indeed, in this framework, we see that space-time can be divided into four disconnected regions, as shown in Fig. 1.6, of the following three distinct types:

$$\begin{aligned}
 (x - y)^2 &= (x - y)_\mu (x - y)^\mu > 0, & \text{time-like,} \\
 (x - y)^2 &= 0, & \text{light-like,} \\
 (x - y)^2 &< 0, & \text{space-like.}
 \end{aligned}
 \tag{1.67}$$

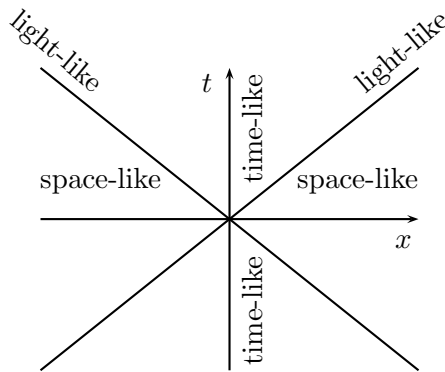


Figure 1.6: Different invariant regions of Minkowski space.

In the time-like regions, the propagation of any signal occurs with a speed less than the speed of light and hence such regions represent causal regions. The light-like regions correspond to regions where signals travel at the speed of light. Finally, in the space-like regions the speed of propagation of any signal exceeds the speed of light and hence, in such regions, no two points can be causally connected. These three types of regions cannot be mapped into one another through a Lorentz transformation. The Einstein view of space-time, therefore, is that at every point in the manifold there exists a cone which corresponds to a causal future cone where the

time evolution of dynamical systems takes place. Furthermore, as is evident from the transformations themselves, the notion of simultaneity does not exist in this manifold. Namely, simultaneity is not a Lorentz invariant concept.

Note that the inhomogeneous Lorentz transformations (Poincaré transformations) also constitute a family (group) of ten parameter transformations (three rotations, three boosts and four space-time translations) – exactly like the Galilean transformations. Therefore, they do not represent a larger symmetry group. However, the nature of the transformations is different. In addition to three dimensional rotations Lorentz transformations also involve boosts. If we consider only the homogeneous Lorentz group we can further restrict ourselves to the proper Lorentz transformations by demanding that

$$\Lambda_0^0 \geq 1, \quad \det \Lambda^\mu_\nu = 1, \quad (1.68)$$

representing orthochronous transformations.

Let us examine how, with these ideas, we can explain the decay of the μ mesons. Let us assume that an observer at rest observes the ticks between his clock to be with intervals $dt = \Delta t$. Therefore, he will calculate the proper length to be

$$ds = (c^2 dt^2 - d\mathbf{x}^2)^{\frac{1}{2}} = c dt \left(1 - \frac{\mathbf{v}^2}{c^2}\right)^{\frac{1}{2}} = c dt = c \Delta t, \quad (1.69)$$

where we have used $\mathbf{v} = 0$ for an observer at rest. However, if another observer traveling with a velocity $\mathbf{v}$ observes the ticks to be of interval $dt' = \Delta t'$, then he will calculate the proper length to be

$$\begin{aligned} ds' &= (c^2 dt'^2 - d\mathbf{x}'^2)^{\frac{1}{2}} = c dt' \left(1 - \frac{\mathbf{v}^2}{c^2}\right)^{\frac{1}{2}} \\ &= c \Delta t' \left(1 - \frac{\mathbf{v}^2}{c^2}\right)^{\frac{1}{2}}. \end{aligned} \quad (1.70)$$

However, since the proper length is an invariant quantity we have

$$ds' = ds,$$

$$\text{or, } \Delta t' = \Delta t \left(1 - \frac{\mathbf{v}^2}{c^2}\right)^{-\frac{1}{2}} = \gamma \Delta t, \quad (1.71)$$

where we have defined

$$\gamma = \left(1 - \frac{\mathbf{v}^2}{c^2}\right)^{-\frac{1}{2}}. \quad (1.72)$$

This shows that the faster an observer is moving, the longer would be the time interval between the ticks of his clock. (The time interval in a clock depends on the speed with which it is moving.) This explains why the μ mesons in cosmic rays have a longer lifetime. The constant γ is conventionally known as the Lorentz factor.

We show next that Maxwell's equations in vacuum without sources are covariant under Lorentz transformations (they are not invariant under Galilean transformations). But before doing that let us introduce the compact notation for four dimensional derivative four vectors (we are going to assume $c = 1$ from now on). The four dimensional covariant derivative is defined as

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \nabla \right), \quad (1.73)$$

which leads to the contravariant derivative

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = \eta^{\mu\nu} \partial_\nu = \left(\frac{\partial}{\partial t}, -\nabla \right). \quad (1.74)$$

Furthermore, we note that

$$\partial_\mu \partial^\mu = \eta^{\mu\nu} \partial_\mu \partial_\nu = \frac{\partial^2}{\partial t^2} - \nabla^2 = \square, \quad (1.75)$$

which is known as the D'Alembertian, is invariant under Lorentz transformations.

To show that Maxwell's equations are Lorentz covariant, let us combine the electric and the magnetic fields into a Lorentz tensor in the following manner. Let us define a second rank anti-symmetric tensor $F_{\mu\nu} = -F_{\nu\mu}$ (it has six independent components in four dimensions) with

$$F_{0i} = E_i \equiv (\mathbf{E})_i, \quad (1.76)$$

and

$$\begin{aligned} F_{ij} &= -\epsilon_{ijk} B_k \equiv -\epsilon_{ijk} (\mathbf{B})_k, \\ \text{or, } B_i &\equiv (\mathbf{B})_i = -\frac{1}{2} \epsilon_{ijk} F_{jk}, \end{aligned} \quad (1.77)$$

where ϵ_{ijk} denotes the three dimensional Levi-Civita tensor. With this identification, two of Maxwell's equations in vacuum without sources (see (1.51)) can be written as

$$\partial^\mu F_{\mu\nu} = 0. \quad (1.78)$$

To see this let us note that this actually reduces to two sets of equations. Namely, for $\nu = 0$, we have ($F_{00} = 0$ by anti-symmetry)

$$\begin{aligned} \partial^\mu F_{\mu 0} &= 0, \\ \text{or, } \partial^i F_{i0} &= 0, \\ \text{or, } (-\nabla) \cdot (-\mathbf{E}) &= \nabla \cdot \mathbf{E} = 0. \end{aligned} \quad (1.79)$$

Similarly, for $\nu = j$, we obtain

$$\begin{aligned} \partial^\mu F_{\mu j} &= 0, \\ \text{or, } \partial^0 F_{0j} + \partial^i F_{ij} &= 0, \\ \text{or, } \frac{\partial E_j}{\partial t} + \partial^i (-\epsilon_{ijk} B_k) &= 0, \\ \text{or, } \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} &= 0, \\ \text{or, } \nabla \times \mathbf{B} &= \frac{\partial \mathbf{E}}{\partial t}. \end{aligned} \quad (1.80)$$

These only represent two of the four equations of Maxwell in (1.51). To obtain the other two, we note that given the field strength tensor $F_{\mu\nu}$, we can define the dual field strength tensor as

$$\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\lambda\rho} F^{\lambda\rho} = \frac{1}{2} \epsilon_{\mu\nu}{}^{\lambda\rho} F_{\lambda\rho}. \quad (1.81)$$

Here $\epsilon^{\mu\nu\lambda\rho}$ (or $\epsilon_{\mu\nu\lambda\rho}$) defines the four dimensional Levi-Civita tensor with $\epsilon^{0123} = 1$. The dual field strength interchanges the roles of the electric and the magnetic fields and we have $\tilde{F}_{0i} = -\frac{1}{2}\epsilon_{ijk}F_{jk} = B_i$ and $\tilde{F}_{ij} = \epsilon_{ijk}E_k$ (where we have used the fact that ϵ^{0ijk} coincides with the three dimensional Levi-Civita tensor ϵ_{ijk}). The other two equations of Maxwell in (1.51) can now be written as

$$\partial^\mu \tilde{F}_{\mu\nu} = 0. \quad (1.82)$$

Equation (1.82) can, in fact, be explicitly solved to determine

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (1.83)$$

where A_μ is known as the (four) vector potential. (It contains the three dimensional scalar and vector potentials as $A^\mu = (\phi, \mathbf{A})$.) Thus, the set of four Maxwell's equations in (1.51) can be written either as

$$\partial^\mu F_{\mu\nu} = 0 = \partial^\mu \tilde{F}_{\mu\nu}, \quad (1.84)$$

or equivalently as

$$\partial^\mu F_{\mu\nu} = 0, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (1.85)$$

Before we examine the Lorentz covariance of Maxwell's equations, let us discuss how various tensors transform under a Lorentz transformation. We already know how a (contravariant) four vector transforms (see (1.61)), namely,

$$x^\mu \rightarrow \Lambda^\mu{}_\nu x^\nu. \quad (1.86)$$

This immediately tells us that under a Lorentz transformation, a second rank contravariant tensor $U^{\mu\nu}$ would transform as

$$U^{\mu\nu} \rightarrow \Lambda^\mu_{\lambda} \Lambda^\nu_{\rho} U^{\lambda\rho}. \quad (1.87)$$

Similarly any higher rank contravariant tensor would transform as

$$U^{\mu\nu\lambda\rho\cdots} \rightarrow \Lambda^\mu_{\mu'} \Lambda^\nu_{\nu'} \Lambda^\lambda_{\lambda'} \Lambda^\rho_{\rho'} \cdots U^{\mu'\nu'\lambda'\rho'\cdots}. \quad (1.88)$$

This is interesting because we can now ask what information can be derived from the transformation property of the invariant (contravariant) metric tensor, namely,

$$\eta^{\mu\nu} \rightarrow \Lambda^\mu_{\lambda} \Lambda^\nu_{\rho} \eta^{\lambda\rho}. \quad (1.89)$$

Since the metric is an invariant tensor, it does not change under a Lorentz transformation. (Transformations which leave the metric of a manifold invariant are also known as isometries of the manifold and Lorentz transformations define isometries of the Minkowski space-time.) This implies

$$\eta^{\mu\nu} \rightarrow \Lambda^\mu_{\lambda} \Lambda^\nu_{\rho} \eta^{\lambda\rho} = \eta^{\mu\nu}, \quad (1.90)$$

which is what we had seen earlier in (1.62). Alternatively, we can write this as

$$\Lambda^\mu_{\lambda} \Lambda^{\nu\lambda} = \eta^{\mu\nu}, \quad \text{or, equivalently} \quad \Lambda^\mu_{\lambda} \Lambda^{\lambda\nu} = \eta^{\mu\nu}. \quad (1.91)$$

Similarly, the inverse (covariant) metric $\eta_{\mu\nu}$ is also an invariant tensor and one can show that this leads to (this also follows from (1.91) by lowering the indices with the metric tensor)

$$\Lambda_{\mu\lambda} \Lambda_\nu^{\lambda} = \eta_{\mu\nu}, \quad \text{or, equivalently} \quad \Lambda_{\lambda\mu} \Lambda_\nu^{\lambda} = \eta_{\mu\nu}. \quad (1.92)$$

Once we know the transformation properties of contravariant tensors as well as the fact that the metric tensor is invariant, then

we can calculate how the lower index (covariant) objects transform under a Lorentz transformation. For example,

$$\begin{aligned} x_\mu &= \eta_{\mu\nu} x^\nu \rightarrow \eta_{\mu\nu} \Lambda^\nu_\lambda x^\lambda \\ &= \Lambda_{\mu\lambda} x^\lambda = \Lambda_\mu^\nu x_\nu. \end{aligned} \quad (1.93)$$

Remembering the relationship (see (1.91))

$$\begin{aligned} \Lambda^\mu_\lambda \Lambda^{\nu\lambda} &= \eta^{\mu\nu}, \\ \text{or, } \Lambda^\mu_\lambda \Lambda_\nu^\lambda &= \delta^\mu_\nu \quad \text{or, equivalently} \quad \Lambda_\lambda^\mu \Lambda^\lambda_\nu = \delta^\mu_\nu, \end{aligned} \quad (1.94)$$

we see that, under a Lorentz transformation, the lower index (covariant) vectors transform in an inverse way compared to the upper index (contravariant) vectors. Therefore, we distinguish between them by saying that

$$\begin{aligned} x^\mu &\rightarrow \text{contravariant vector,} \\ x_\mu &\rightarrow \text{covariant vector,} \\ U^{\mu\nu\lambda\rho\dots} &\rightarrow \text{contravariant tensor,} \\ U_{\mu\nu\lambda\rho\dots} &\rightarrow \text{covariant tensor.} \end{aligned} \quad (1.95)$$

We are now in a position to calculate how a covariant tensor transforms under a Lorentz transformation

$$\begin{aligned} U_{\mu\nu} &\rightarrow \Lambda_\mu^\lambda \Lambda_\nu^\rho U_{\lambda\rho}, \\ U_{\mu\nu\lambda\rho\dots} &\rightarrow \Lambda_\mu^{\mu'} \Lambda_\nu^{\nu'} \Lambda_\lambda^{\lambda'} \Lambda_\rho^{\rho'} \dots U_{\mu'\nu'\lambda'\rho'\dots} \end{aligned} \quad (1.96)$$

In addition, we see that we can have tensors of mixed nature. A tensor with m contravariant indices and n covariant ones would transform under a Lorentz transformation as

$$U^{\mu_1\dots\mu_m}_{\nu_1\dots\nu_n} \rightarrow \Lambda^{\mu_1}_{\mu'_1} \dots \Lambda^{\mu_m}_{\mu'_m} \Lambda_{\nu_1}^{\nu'_1} \dots \Lambda_{\nu_n}^{\nu'_n} U^{\mu'_1\dots\mu'_m}_{\nu'_1\dots\nu'_n} \quad (1.97)$$

We also see now that the length of a vector transforms as

$$\begin{aligned}
 x^2 = x_\mu x^\mu &\rightarrow \Lambda_\mu^\nu x_\nu \Lambda_\lambda^\mu x^\lambda \\
 &= \Lambda_\mu^\nu \Lambda_\lambda^\mu x_\nu x^\lambda \\
 &= \delta_\lambda^\nu x_\nu x^\lambda = x_\nu x^\nu = x^2,
 \end{aligned} \tag{1.98}$$

where we have used (1.94). Namely, the length of a four vector is a Lorentz scalar as we have observed earlier.

Furthermore, let us note that since

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu, \tag{1.99}$$

we can write

$$x^\mu = \Lambda^\mu_\nu x'^\nu, \tag{1.100}$$

which follows from (1.94), namely,

$$\begin{aligned}
 x'^\mu &= \Lambda^\mu_\nu x^\nu, \\
 \text{or, } \Lambda_\mu^\lambda x'^\mu &= \Lambda_\mu^\lambda \Lambda^\mu_\nu x^\nu = \delta_\nu^\lambda x^\nu = x^\lambda, \\
 \text{or, } x^\lambda &= \Lambda_\mu^\lambda x'^\mu, \\
 \text{or, } x^\mu &= \Lambda_\nu^\mu x'^\nu.
 \end{aligned} \tag{1.101}$$

Similarly, since

$$x_\mu \rightarrow x'_\mu = \Lambda_\mu^\nu x_\nu, \tag{1.102}$$

using (1.94), it follows that

$$x_\mu = \Lambda_\mu^\nu x'_\nu. \tag{1.103}$$

We are now in a position to calculate how the four derivatives transform under a Lorentz transformation. For example,

$$\begin{aligned}
\partial^\mu &= \frac{\partial}{\partial x_\mu} \rightarrow \frac{\partial}{\partial x'_\mu} = \frac{\partial x_\nu}{\partial x'_\mu} \frac{\partial}{\partial x_\nu} \\
&= \Lambda^\mu{}_\nu \frac{\partial}{\partial x_\nu} = \Lambda^\mu{}_\nu \partial^\nu,
\end{aligned} \tag{1.104}$$

where we have used (1.103). Similarly, using (1.100) we obtain

$$\begin{aligned}
\partial_\mu &= \frac{\partial}{\partial x^\mu} \rightarrow \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} \\
&= \Lambda_\mu{}^\nu \frac{\partial}{\partial x^\nu} = \Lambda_\mu{}^\nu \partial_\nu.
\end{aligned} \tag{1.105}$$

Thus we see that contragradient and the cgradient vectors transform like contravariant and covariant vectors respectively. From these, it now follows that the D'Alembertian (1.75) is invariant under a Lorentz transformation.

We are now in a position to discuss the Lorentz covariance of Maxwell's equations. First of all we note that since the field strengths are tensors of second rank, under a Lorentz transformation, they transform as (see (1.96))

$$\begin{aligned}
F_{\mu\nu} &\rightarrow \Lambda_\mu{}^\lambda \Lambda_\nu{}^\rho F_{\lambda\rho}, \\
\tilde{F}_{\mu\nu} &\rightarrow \Lambda_\mu{}^\lambda \Lambda_\nu{}^\rho \tilde{F}_{\lambda\rho}.
\end{aligned} \tag{1.106}$$

Thus,

$$\begin{aligned}
\partial^\mu F_{\mu\nu} &\rightarrow \partial'^\mu F'_{\mu\nu} = \Lambda^\mu{}_{\mu'} \partial^{\mu'} \Lambda_\mu{}^\lambda \Lambda_\nu{}^\rho F_{\lambda\rho} \\
&= \Lambda^\mu{}_{\mu'} \Lambda_\mu{}^\lambda \Lambda_\nu{}^\rho \partial^{\mu'} F_{\lambda\rho} \\
&= \delta_{\mu'}^\lambda \Lambda_\nu{}^\rho \partial^{\mu'} F_{\lambda\rho} = \Lambda_\nu{}^\rho \partial^\lambda F_{\lambda\rho}.
\end{aligned} \tag{1.107}$$

This shows that if

$$\partial^\mu F_{\mu\nu} = 0, \tag{1.108}$$

in one inertial frame of reference, the form of the equation does not change in a Lorentz transformed frame of reference. Similarly, we can also show that the equation for the dual (1.82)

$$\partial^\mu \tilde{F}_{\mu\nu} = 0, \quad (1.109)$$

also maintains its form in any inertial Lorentz frame so that Maxwell's equations are Lorentz covariant. We should note here that since

$$F_{\mu\nu} \rightarrow \Lambda_\mu^\lambda \Lambda_\nu^\rho F_{\lambda\rho}, \quad (1.110)$$

and the transformation matrices Λ_μ^λ are not necessarily diagonal, Lorentz transformations (boosts) mix up different field components. This is another way of saying that under a Lorentz transformation, the electric and the magnetic fields mix with each other which we are all familiar with.

► **Example (Null basis vectors).** As we have seen in (1.67), in Minkowski space vectors can be time-like, space-like or light-like (null). Let us ask if it is possible to find four linearly independent null (light-like) vectors in four dimensional Minkowski space which are also orthogonal.

It is quite easy to find four linearly independent null four vectors. For example, we can have

$$\begin{aligned} e_1^\mu &= (1, 1, 0, 0), & e_2^\mu &= (1, 0, 1, 0), \\ e_3^\mu &= (1, 0, 0, 1), & e_4^\mu &= (1, -1, 0, 0). \end{aligned} \quad (1.111)$$

This set is obviously linearly independent and each vector is easily checked to be light-like (null), namely, $\eta_{\mu\nu} e_i^\mu e_i^\nu = 0, i$ (not summed) $= 1, 2, 3, 4$. Such a set defines a basis for the Minkowski space. In fact, this is the light-cone basis for a photon. Note, however, that these basis vectors are not orthogonal.

It is not possible to find a set which is in addition orthogonal and null for the following reason. We will attempt to build an orthogonal basis out of a set of null vectors via the Gram-Schmidt procedure. To that end, take any of the null vector and label it v_1^μ . v_1^μ necessarily has a time component and a space component (it is the only way it can be null but non-zero) so we can write

$$v_1^\mu = (1, \mathbf{v}_1), \quad \mathbf{v}_1 \cdot \mathbf{v}_1 = 1. \quad (1.112)$$

To get the second vector v_2^μ we know it, too, must have a time and space component so up to a multiplicative constant we can write

$$v_2^\mu = (1, \mathbf{v}_2), \quad \mathbf{v}_2 \cdot \mathbf{v}_2 = 1. \quad (1.113)$$

If we now impose orthogonality between the two vectors (1.112) and (1.113), we obtain

$$v_1 \cdot v_2 = 1 - \mathbf{v}_1 \cdot \mathbf{v}_2 = 0, \quad (1.114)$$

which implies

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 1. \quad (1.115)$$

But, $\mathbf{v}_1, \mathbf{v}_2$ are (three dimensional) unit vectors so that (1.115) implies that $\mathbf{v}_1 = \mathbf{v}_2$ which would amount to their not being linearly independent. This proves that it is impossible to find a set of four linearly independent light-like (null) vectors in Minkowski space which are also orthogonal.



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Relativistic dynamics

2.1 Relativistic point particle

It is clear that Newton's equation which is covariant (form invariant) under Galilean transformations, is not covariant under Lorentz transformations. So if we require Lorentz invariance to be the invariance group of the physical world, we must modify Newton's laws. Let us note, however, that since Newton's laws have been tested again and again in the laboratory for slowly moving particles, it is only for highly relativistic particles ($|\mathbf{v}| \simeq c = 1$) that the equation may require any modification.

We have already seen that in Minkowski space we can write the line element as

$$ds^2 = d\tau^2 = \eta_{\mu\nu} dx^\mu dx^\nu = dt^2 - d\mathbf{x}^2, \quad (2.1)$$

where $d\tau$ is known as the “proper time” of the particle and is related to the proper length through the speed of light ($ds = cd\tau$) which we have set to unity. The proper time, like the proper length, is an invariant quantity (since the speed of light is an invariant) and hence can be used to characterize the trajectory of a particle. To describe a Lorentz covariant generalization of Newton's equation, let us define a force four vector f^μ such that

$$m \frac{d^2 x^\mu}{d\tau^2} = f^\mu, \quad (2.2)$$

where m represents the rest mass of the particle. Clearly since the proper time does not change under a Lorentz transformation, equation (2.2) takes the same form in all Lorentz frames. Furthermore, if

we know the form of f^μ , we can calculate the trajectory of the particle. To understand the nature of this force four vector, we note that the form of the relativistic force four vector can be obtained from the observation that since (2.2) holds in all Lorentz frames, it must be valid in the rest frame as well. In the rest frame of the particle ($\frac{d\mathbf{x}}{dt} = 0$)

$$d\tau = dt, \quad (2.3)$$

and hence (2.2) takes the form

$$\begin{aligned} m \frac{d^2 x^i}{dt^2} &= f_{(\text{rest})}^i, \\ f_{(\text{rest})}^0 &= 0. \end{aligned} \quad (2.4)$$

Furthermore, if we define the relativistic four velocity of a particle as

$$u^\mu = \frac{dx^\mu}{d\tau} = \left(\frac{dt}{d\tau}, \frac{d\mathbf{x}}{d\tau} \right) = \frac{dt}{d\tau} \left(1, \frac{d\mathbf{x}}{dt} \right) = \gamma(1, \mathbf{v}), \quad (2.5)$$

then in the rest frame of the particle we have $u_{(\text{rest})}^\mu = (1, 0, 0, 0)$. Therefore, in the rest frame we obtain

$$f_{(\text{rest})}^\mu u_{(\text{rest})}^\nu \eta_{\mu\nu} = f_{(\text{rest})}^\mu u_{\mu}^{(\text{rest})} = 0. \quad (2.6)$$

On the other hand, this is a (Lorentz invariant) scalar equation and hence must be true in any inertial reference frame. Secondly, we see explicitly that

$$u_{(\text{rest})}^\mu u_{(\text{rest})}^\nu \eta_{\mu\nu} = u_{(\text{rest})}^\mu u_{\mu}^{(\text{rest})} = 1. \quad (2.7)$$

This must also remain true in all inertial reference frames as can be seen from (2.5) using the definition of the Lorentz factor in (1.72). That these two equations are compatible can be seen in the following way. If

$$u^\mu u^\nu \eta_{\mu\nu} = 1, \quad (2.8)$$

it follows that

$$\begin{aligned} \frac{d}{d\tau} (u^\mu u^\nu \eta_{\mu\nu}) &= 0, \\ \text{or, } \frac{du^\mu}{d\tau} u^\nu \eta_{\mu\nu} &= 0, \\ \text{or, } \frac{1}{m} f^\mu u^\nu \eta_{\mu\nu} &= 0, \\ \text{or, } f^\mu u^\nu \eta_{\mu\nu} &= f^\mu u_\mu = 0. \end{aligned} \quad (2.9)$$

Here we have used the fact that

$$f^\mu = m \frac{d^2 x^\mu}{d\tau^2} = m \frac{du^\mu}{d\tau}. \quad (2.10)$$

Equation (2.9) shows that the relativistic force (acceleration) has to be orthogonal to the four velocity and, as a result, not all components of the force four vector are independent.

In analogy with the non-relativistic case, we can define the four momentum of a particle as

$$p^\mu = m u^\mu, \quad (2.11)$$

so that the dynamical equation (2.2) (or (2.10)) can also be written in the familiar form (the rest mass m is a constant)

$$\frac{dp^\mu}{d\tau} = f^\mu. \quad (2.12)$$

We note that the space components of the four velocity are given by (see (2.5))

$$\mathbf{u} = \frac{d\mathbf{x}}{d\tau} = \gamma \mathbf{v}, \quad (2.13)$$