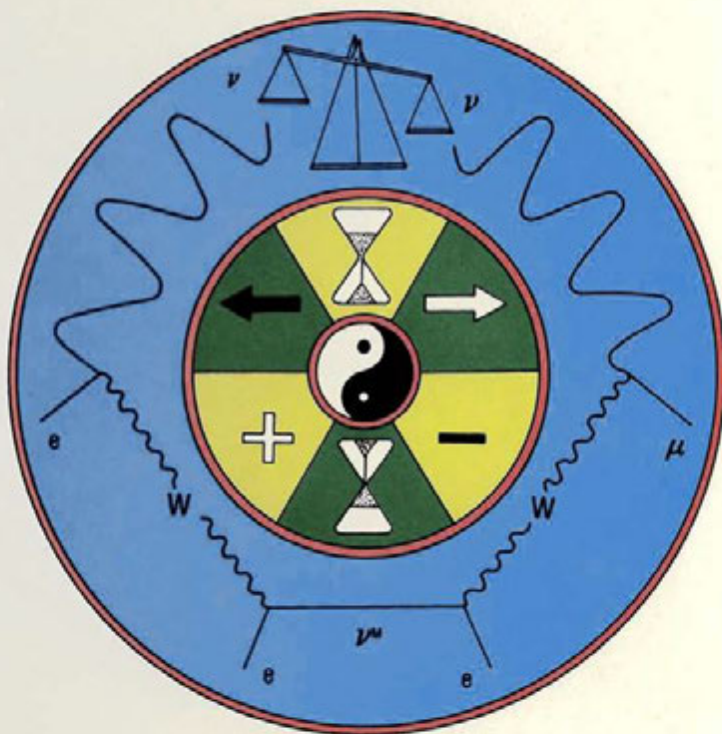


World Scientific Lecture Notes in Physics Vol. 25

# THE PHYSICS OF MASSIVE NEUTRINOS



**BORIS KAYSER**  
with  
**FRANÇOISE GIBRAT-DEBU and FREDERIC PERRIER**

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World Scientific Lecture Notes in Physics Vol. 25

# THE PHYSICS OF MASSIVE NEUTRINOS

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# PREFACE

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These lectures, given originally at Saclay in November 1985, are intended to explain the physics and phenomenology of massive neutrinos. They require of the reader only a knowledge of quantum mechanics and of very elementary quantum field theory.

After arguing that neutrino mass is not unlikely, and briefly considering the search for evidence of this mass in decay processes, we examine the physics and phenomenology of neutrino oscillation. Then we discuss the physics of Majorana neutrinos (neutrinos which are their own antiparticles). We first establish a number of basic properties of such neutrinos without using any field theory. We then introduce the field-theory description of a Majorana particle, and use it to treat such processes as neutrinoless double beta decay. Finally, having studied Majorana neutrinos, we turn to the treatment of neutrino masses in gauge theories, and derive the “see-saw relation”, which explains why neutrino masses are so small by relating them to the inverse of a large mass scale.

Our references to experimental results and to the literature were compiled in March 1987, and updated in July 1988.

*Boris Kayser  
Frédéric Perrier  
Françoise Gibrat-Debu*

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# **THE PHYSICS OF MASSIVE NEUTRINOS**

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## INTRODUCTION

Why should we talk about neutrino masses? There is no incontrovertible experimental evidence that neutrinos have mass. Nevertheless, some physicists believe that they are in fact massive.

A good reason for this belief is that from the standpoint of the grand unified theories (GUTS), which try to unify the description of the weak, electromagnetic, and strong interactions, it is more natural for neutrinos to be massive than to be massless. In any grand unified theory, we put a given neutrino  $\nu$  in a large multiplet  $F$  together with at least one charged lepton  $\ell$ , one positively-charged quark  $q^+$ , and one negatively-charged quark  $q^-$ :

$$F = \begin{bmatrix} q^+ \\ q^- \\ \ell \\ \nu \end{bmatrix}$$

In this scheme, neutrinos become brothers of leptons and quarks which are all massive. Thus, it would be exceptional for them to be massless! Of course, this is not a proof that neutrinos have mass. However, it does make it very natural for them to do so.

Here we shall assume that neutrinos do indeed have mass. To be sure, there is as yet no very good evidence for the grand unified theories.

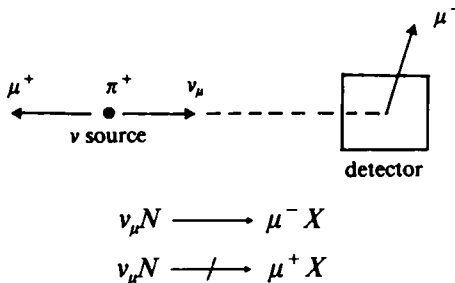
Proton decay, which is perhaps the most spectacular prediction of these models, has not been observed, and the proton lifetime predicted by the minimal SU(5) GUT is ruled out by experiment. However, this particular GUT is a somewhat unnatural one, since it does not place all the quarks and leptons of one generation in a single multiplet, but in a package consisting of two multiplets. Grand unified theories as a class are very much alive. In fact, the famous and popular Superstring Theories strongly suggest that, at the appropriate energy scale, Nature is indeed described by a grand unified theory.

Let us return to the multiplet  $F$ . We know experimentally that the neutrino mass is much smaller than the quark and lepton masses:

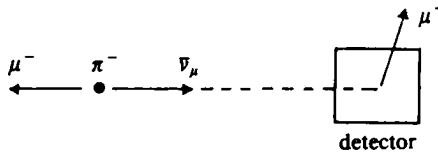
$$M_\nu \ll M_{q^-}, M_{q^+}, M_\ell$$

How does the neutrino get to be much lighter than all the other fermions? Perhaps, unlike  $q^+$ ,  $q^-$ ,  $\ell$ , which carry electric charge, the neutrino is its own antiparticle, and perhaps this circumstance is the origin of its relative lightness.

But what about the lepton number? Does not this quantum number distinguish a  $\bar{\nu}$  from a  $\nu$ ? To see why it may not do that, let us recall the origin of the introduction of this number. It is observed experimentally that the neutral particle which is emitted simultaneously with a  $\mu^+$  in the  $\pi^+$  decay interacts with matter producing only a  $\mu^-$  and never a  $\mu^+$ . This neutral particle is called  $\nu_\mu$ .



It is similar for the  $\pi^-$  decay: the neutral particle called  $\bar{\nu}_\mu$  which is emitted simultaneously with the  $\mu^-$  interacts with matter producing only a  $\mu^+$  and never a  $\mu^-$ :



$$\bar{\nu}_\mu N \longrightarrow \mu^+ X$$

$$\bar{\nu}_\mu N \not\longrightarrow \mu^- X$$

The conventional explanation for these facts is the following:

- (i)  $\nu_\mu$  and  $\bar{\nu}_\mu$  are distinct from each other.
- (ii) There is a quantum number which is conserved during these interactions: the lepton number. We assign the lepton number  $+1$  to  $\mu^-$  and  $\nu_\mu$  and  $-1$  to  $\mu^+$  and  $\bar{\nu}_\mu$ ; then  $\nu_\mu$  can only produce  $\mu^-$  in matter.

However, there is no evidence that this explanation is the correct one. Indeed, the existence of the lepton number is not necessary.

An alternative explanation arises from the fact that the neutral particle produced in the  $\pi^+$  decay has a left-handed helicity, while the neutral particle produced in the  $\pi^-$  decay has a right-handed helicity. Now, suppose that the parity-violating weak interaction is such that every neutrino with a right-handed helicity interacts giving a  $\mu^+$ , while every neutrino with a left-handed helicity interacts giving a  $\mu^-$ . In this case we don't need to introduce a conserved lepton number! The helicity constraints are sufficient to account for all the observed facts. If they are the true explanation, then, in contradiction with (i), “ $\nu_\mu$ ” and “ $\bar{\nu}_\mu$ ” are, respectively, just the left and right helicity states of a single particle  $\nu^M$  (M for Majorana). In contrast, if the hypothesis (i)–(ii) is the true explanation, then  $\nu_\mu$  is a so-called Dirac neutrino  $\nu^D$ . This is a particle with two helicity states whose antiparticle  $\bar{\nu}^D$  (here  $\bar{\nu}_\mu$ ) differs from  $\nu^D$  and has two helicity states of its own, for a total of four states. We will explain the difference between Dirac and Majorana neutrinos in much greater detail later.

To tell whether “ $\nu_\mu$ ” and “ $\bar{\nu}_\mu$ ” are just the two helicity states of a Majorana neutrino, or do truly differ in some way other than helicity, one would like to carry out the following experiment: Produce left-handed  $\nu_\mu$  particles in  $\pi^+$  decays. Somehow reverse the helicities of these particles, turning them into right-handed objects. Now find out whether these right-handed objects interact with matter in the same way as the right-handed “ $\bar{\nu}_\mu$ ” particles produced in  $\pi^-$  decays (Majorana case), or not (Dirac case). Unfortunately, this experiment would be extremely difficult, and it has not been done.

In the most popular explanation of the lightness of neutrinos relative to the charged leptons and quarks, a particle which starts out as a Dirac neutrino (four states:  $\nu^D$  and  $\bar{\nu}^D$ , each with two helicities) splits into two Majorana neutrinos (two states each). One of these Majorana neutrinos,  $\nu$ , is light, and is identified as the familiar neutrino, while the other one,  $N$ , is heavy. In this scheme, which is due to Gell-Mann-Ramond-Slansky, Yanagida, and Mohapatra-Senjanovic,<sup>1</sup> it is natural for the  $\nu$  and  $N$  masses to be related by

$$M_\nu M_N = M_{q \text{ or } \ell}^2 \quad (1.1)$$

where  $M_{q \text{ or } \ell}$  is a typical quark or charged lepton mass. Furthermore,  $M_N$  is related to a symmetry-breaking scale, and so is in general quite big. Thus we can conclude that  $M_\nu \ll M_{q \text{ or } \ell}$ .

Using relation (1.1), we can get some orders of magnitude: if  $M_N \sim M_{\text{GUT}} = 10^{14}$  GeV and  $M_{q \text{ or } \ell} \sim 1$  GeV then  $M_\nu \sim 10^{-5}$  eV! We can have neutrino masses much smaller than what can now be measured! If  $M_N \sim 100$  GeV and  $M_{q \text{ or } \ell} \sim 1$  MeV we obtain a mass  $M_\nu$  of the order of magnitude of the experimental limits.

Recall that the mass of the neutrino has fundamental consequences in astrophysics: if the mass of the neutrino is of order of 10 eV and if its lifetime is large compared to the age of the universe, the neutrinos could dominate the mass density of the universe.

We would like to end this introduction by emphasizing the importance of trying to find new experiments that would be sensitive to neutrino masses well below 1 eV!

# PHENOMENOLOGY OF MASSIVE NEUTRINOS

Now, we would like to go through some phenomenological consequences of neutrino mass, focussing on how one can look for evidence of it. To make things clear, let us define the following simple theoretical framework.

## 1 – A SIMPLE THEORETICAL FRAMEWORK

Assume that we have three charged leptons,  $e$ ,  $\mu$  and  $\tau$ . Each of these leptons  $\ell$  is coupled to a neutral state called  $\nu_\ell$  (neutrino associated with the lepton  $\ell$ ) via the intermediate vector boson discovered at CERN with a mass around 83 GeV. The Hamiltonian density for the charged current weak interaction is then ( $g$  is a coupling constant):

$$H_{CC} = gW_\mu(83 \text{ GeV}) \sum_{\ell=e,\mu,\tau} i\bar{\ell}\gamma_\mu(1 + \gamma_5)\nu_\ell + h. c. \quad (2.1)$$

The set  $\{\nu_\ell\}$  is said to be the set of flavor eigenstates or weak interaction eigenstates.

Now assume that the freely propagating neutrinos of definite mass are not the  $\nu_\ell$  but a set of three neutral states  $\nu_m$  with masses  $M_m$ , and that the flavor eigenstates  $\nu_\ell$  are linear combinations of these mass eigenstates  $\nu_m$ :

$$\nu_\ell = \sum_m U_{\ell m} \nu_m \quad (2.2)$$

Here  $U$  is the neutrino mixing matrix. We have

$$H_{CC} = gW_\mu(83 \text{ GeV}) \sum_{\ell=e,\mu,\tau} i\bar{\ell}\gamma_\mu(1 + \gamma_5) \sum_m U_{\ell m} \nu_m + h.c.$$

The mixing matrix  $U$  is the analog of the Kobayashi-Maskawa matrix in the charged current Hamiltonian between quarks:

$$H_{CC} = gW_\mu(83 \text{ GeV}) \sum_i i\bar{q}_i^+ \gamma_\mu(1 + \gamma_5) \sum_j (KM)_{ij} q_j^- + h.c. \quad (2.3)$$

where:  $i$  and  $j$  are generation indices  
 $q^+$  is a positive quark  $u, c, \text{ or } t$   
 $q^-$  is a negative quark  $d, s, \text{ or } b$   
 $KM$  is the Kobayashi-Maskawa matrix

The existence of the quark mixing, described by the Kobayashi-Maskawa matrix, has been established experimentally. It is then natural to expect that there is lepton mixing, described by the matrix  $U$ , as well. The mixing matrix  $U$  has the following properties:

(i) In a gauge theory, it is a unitary matrix: it relates the basis  $\{\nu_m\}$  of the eigenstates of the free-particle Hamiltonian to the basis  $\{\nu_\ell\}$  of the eigenstates of the gauge group.

(ii) If  $CP$  is conserved in our theory, the matrix  $U$  can be chosen real. We will explain this later on.

Let us now describe simple phenomena involving massive neutrinos and firstly: decays.

## 2 – WHAT WE HAVE LEARNT FROM DECAYS

Consider a decay involving a neutrino in the final state:

$$X \rightarrow Y + \nu$$

Due to the neutrino mixing, this decay is actually the incoherent sum of decays into all the neutrino mass eigenstates whose emission is kinematically possible:

$$X \rightarrow Y + \nu = \sum_m (X \rightarrow Y + \nu_m)$$

For example, the decay  $\pi \rightarrow \mu \nu_\mu$  is actually:

$$\pi \rightarrow \mu + \nu_\mu = \sum_m (\pi \rightarrow \mu + \nu_m)$$

The spectrum of the final muon momentum will then have several monochromatic lines, each of these corresponding to a mass eigenstate  $\nu_m$ , and having a strength proportional to  $|U_{\mu m}|^2$  (Ref. 2). People have looked at this spectrum, and seen no lines. Taking into account the experimental resolution, they have obtained limits on  $|U_{\mu m}|^2$  for  $\nu_m$  masses above  $\sim 10$  MeV.<sup>3</sup> In addition, study of the decay  $\pi \rightarrow \mu + \nu_\mu$  has shown that the mass of the eigenstate  $\nu_m$  which is emitted most of the time in the decay is less than 250 keV.<sup>4</sup> This statement is shortened to:

$$M(\nu_\mu) < 250 \text{ keV}$$

Information on the mass eigenstate which is the dominant piece of the flavor eigenstate  $\nu_e$  is coming from experiments on tritium  $\beta$  decay:

$$\text{T} \rightarrow {}^3\text{He} + e^- + \bar{\nu}_e$$

An experiment in Moscow claims that the dominant mass eigenstate in this decay has a mass between 17 and 40 eV.<sup>5</sup> But this result is quite controversial. A number of other experiments on tritium  $\beta$  decay are in progress.<sup>6,7</sup>

Limits on the  $\tau$  neutrino mass, the mass of the eigenstate  $\nu_m$  mostly coupled to the  $\tau$  lepton, have been obtained from the decay of  $\tau$  leptons produced in  $e^+e^-$  collisions. The most stringent limit comes from study of the  $5\pi$  invariant mass distribution in the reaction:

$$e^+e^- \rightarrow \tau^+\tau^- \rightarrow (5\pi)^- \nu_\tau$$

The result is (Ref. 8):

$$M(\nu_i) < 35 \text{ MeV}$$

In all the processes above, the distinction between Majorana and Dirac neutrinos is irrelevant.

A very interesting process is the rare decay of the kaon into one pion and two neutrinos (this decay is not yet observed):<sup>9</sup>

$$K^+ \rightarrow \pi^+ \nu \bar{\nu} = \sum_m (K^+ \rightarrow \pi^+ \nu_m \bar{\nu}_m)$$

This decay has the very special property that it is, in principle, sensitive to the Dirac or Majorana character of the neutrinos.<sup>10</sup>

The only thing that can be measured here is the energy of the outgoing pion. One can show that, due to the unitarity of the  $U$  matrix, the pion energy spectrum does not depend on the  $U_{lm}$ . However, as shown in Figure 1,<sup>10</sup> this spectrum *does* depend on the neutrino mass (we assume only one  $\nu_m$  has a mass that is significant). In addition to this, if the emitted neutrinos are nonrelativistic (if the mass of the heaviest mass eigenstate is large enough), the distortion of the spectrum is very different if these neutrinos are Dirac or Majorana particles (see Figure 1(a)). If, however, the neutrinos are relativistic (neutrino masses very small compared to a typical momentum), it becomes very hard to tell whether they are Majorana or Dirac particles (see Figure 1(b)).

This is, in fact, an example of a very general phenomenon which we refer to as the Majorana-Dirac confusion theorem:<sup>11</sup> Assuming there are no right-handed currents, if the neutrino masses are very small compared to the experimental energy scale, it is impossible to tell the difference between a Dirac neutrino and a Majorana neutrino. We shall come back to this theorem later.

Besides decays, a very important consequence of the neutrino masses is the possible existence of neutrino oscillations.

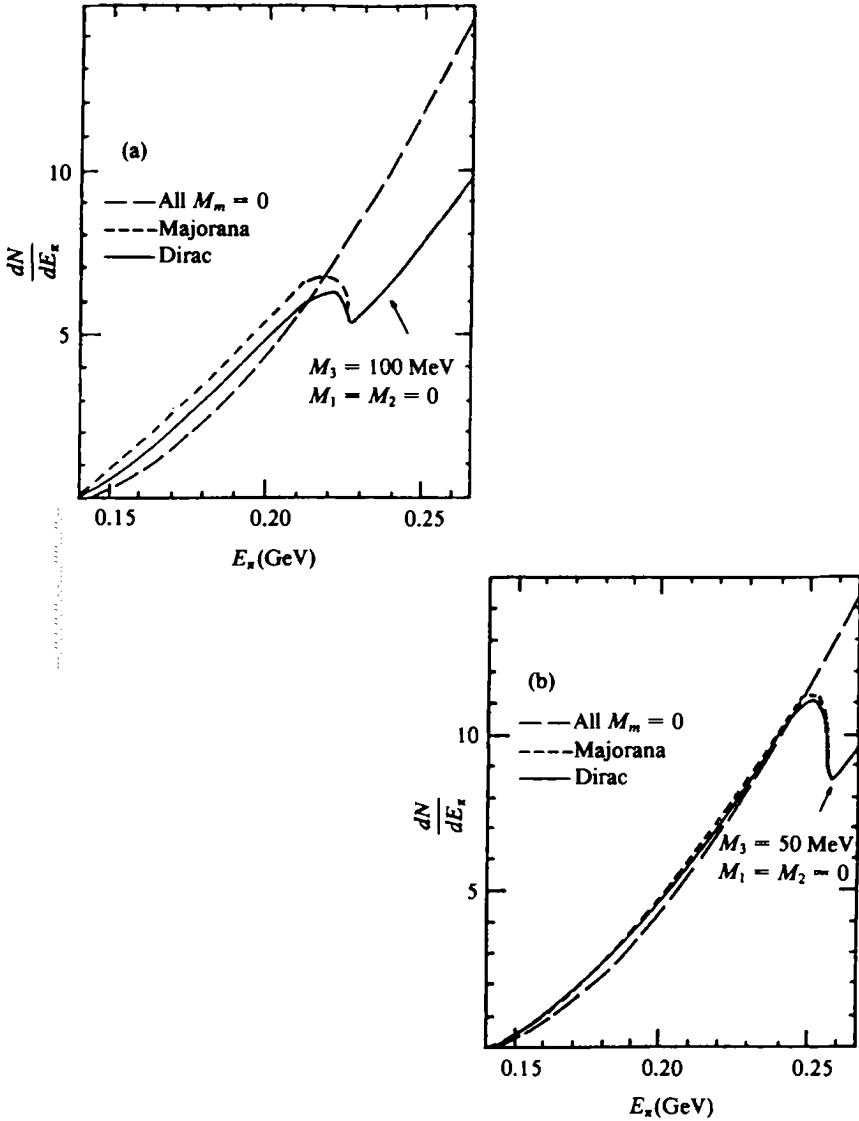


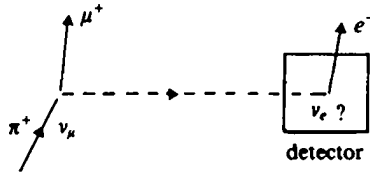
Fig. 1. Effect of the neutrino mass on the decay  $K \rightarrow \pi \nu \bar{\nu}$ .

### 3 – THE NEUTRINO OSCILLATIONS

#### a) Neutrino oscillations in vacuum

##### i) Principle of neutrino oscillations

Let us describe a typical neutrino-oscillation experiment. Take a pion beam and let the pions decay into muons and neutrinos (by definition, see chapter II.1, these neutrinos are called muon neutrinos  $\nu_\mu$ ). Downstream of the decaying pions is a target-detector looking for neutrino interactions. Suppose that we see an interaction with an *electron* in the final state:



The trouble is that the neutrino which is able to produce an electron is not  $\nu_\mu$  but  $\nu_e$ . The only explanation is that our muon neutrino  $\nu_\mu$  has transformed itself into an electron neutrino  $\nu_e$  between the decay point and the detector. How is that possible?

The answer is given by quantum mechanics. Assume, to be quite general, that we have  $N$  flavors of charged leptons  $\ell = e, \mu, \tau, \dots$  associated with  $N$  neutrinos  $\nu_e, \nu_\mu, \nu_\tau, \dots$ . Suppose that we have a source of  $\nu_\ell$  neutrinos. We said that the flavor eigenstate  $\nu_\ell$  is not a mass eigenstate but a linear superposition of mass eigenstates  $\nu_m$ . Assume that there are  $N$   $\nu_m$ , and that the mass resolution of the experiment is much poorer than the level of sensitivity,  $M_m - M_{m'}$ , needed to distinguish one  $\nu_m$  from another. Then each neutrino produced by our source is the coherent superposition:

$$\nu_\ell = \sum_{m=1}^N U_{\ell m} \nu_m$$

of the mass eigenstates.

Let us assume for the sake of simplicity that we have a neutrino  $\nu_\ell$  born

at time  $t = 0$  with momentum  $p$ , perfectly defined. At this time, the wave function is therefore:

$$\psi(x, t = 0) = \sum_m U_{\ell m} v_m e^{ip_\nu x} \quad (2.4)$$

After a time  $t$ , this will evolve into:

$$\psi(x, t) = \sum_m U_{\ell m} v_m e^{ip_\nu x} e^{-iE_m t},$$

with

$$E_m = E(v_m) = \sqrt{p_\nu^2 + M_m^2}$$

Assuming  $M_m \ll p_\nu$ ,

$$E_m \simeq p_\nu + \frac{M_m^2}{2p_\nu}$$

With all  $M_m \ll p_\nu$ , our  $\nu$  is travelling almost with the speed of light. Hence, if it was born at  $x = 0$ , then at time  $t$ , it will be approximately at  $x = t$ . Let us then examine  $\psi(t, t) = \psi(x, x)$ , which is given by:

$$\psi(x, x) \simeq \sum_m U_{\ell m} v_m e^{-i[M_m^2/2p_\nu]x}$$

Expressing  $v_m$  as a combination of the  $v_{\ell'}$ 's, we find:

$$v_m = \sum_{\ell'} U_{\ell' m}^* v_{\ell'}$$

so:

$$\psi(x, x) = \sum_{\ell'} \left[ \sum_m U_{\ell m} e^{-i(M_m^2/2p_\nu)x} U_{\ell' m}^* \right] v_{\ell'}. \quad (2.5)$$

We see that this wave function is a superposition of all the neutrino flavors. In it, the amplitude for our neutrino, born with flavor  $\ell$ , to have a new flavor  $\ell'$  after travelling a distance  $x$  is just the coefficient of  $v_{\ell'}$ .

Note that, as illustrated in Figure 2, this amplitude has the same structure as that for a multi-slit interference experiment. It consists of an amplitude for the original  $\nu_\ell$  to be a  $\nu_m$ , times an amplitude for the  $\nu_m$ , having propagated, to be a  $\nu_{\ell'}$ . This product of amplitudes is then summed coherently over all possible “slits”  $\nu_m$ .

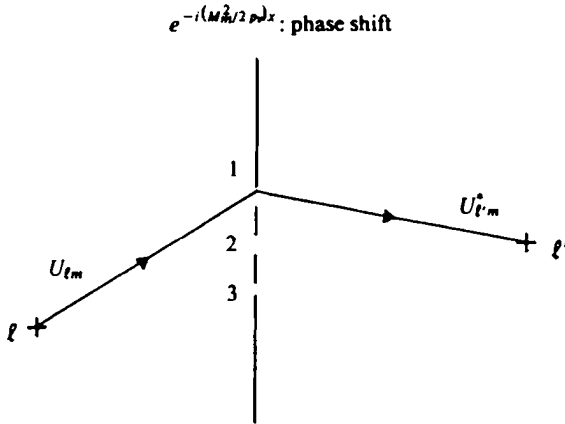


Fig. 2. Multi-slit interference experiment. The structure of the amplitude of the neutrino oscillation is very general in quantum mechanics.

Neutrino oscillation can be understood in a very simple way. What happens is that, at a given  $p_\nu$ , the lighter mass states in the original  $\nu_\ell$  travel faster than the heavier ones, and get ahead of the latter. Thus, the various  $\nu_m$  components of the beam get out of phase with one another, and do not add up to a  $\nu_\ell$  anymore. Thus, as it travels, the beam picks up components corresponding to other flavors.

Let  $P(\ell \rightarrow \ell', x)$  be the probability of finding the neutrino to have flavor  $\ell'$  at a distance  $x$  from its source, if originally it had flavor  $\ell$ . From Eq. (2.5), we have:

$$\begin{aligned}
P(\ell \rightarrow \ell', x) &= \left[ \sum_m U_{\ell m}^* e^{iM_m^2 x/2p_\nu} U_{\ell' m} \right] \cdot \left[ \sum_m U_{\ell m} e^{-iM_m^2 x/2p_\nu} U_{\ell' m}^* \right] \\
&= \sum_m |U_{\ell m}|^2 |U_{\ell' m}|^2 \\
&\quad + \sum_{m' \neq m} \text{Re}(U_{\ell m} U_{\ell m'}^* U_{\ell' m'} U_{\ell' m}^*) \cos\left(\frac{M_m^2 - M_{m'}^2}{2p_\nu} x\right) \\
&\quad + \sum_{m' \neq m} \text{Im}(U_{\ell m} U_{\ell m'}^* U_{\ell' m'} U_{\ell' m}^*) \sin\left(\frac{M_m^2 - M_{m'}^2}{2p_\nu} x\right)
\end{aligned} \tag{2.6}$$

Now, we make the assumption that *CP is conserved*: one can then choose *U real*, and the previous expression simplifies to:

$$P(\ell \rightarrow \ell', x) = \sum_m U_{\ell m}^2 U_{\ell' m}^2 + \sum_{m' \neq m} U_{\ell m} U_{\ell m'} U_{\ell' m'} U_{\ell' m} \cos\left(2\pi \frac{x}{L_{mm'}}\right) \tag{2.7}$$

This probability has a beautiful oscillatory pattern as a function of distance  $x$ . In it, the quantities  $L_{mm'}$  are:

$$L_{mm'} = 2\pi \frac{2p_\nu}{|M_m^2 - M_{m'}^2|} \equiv 2\pi \frac{2p_\nu}{\delta M_{mm'}^2} \tag{2.8}$$

$L_{mm'}$  is the *oscillation length* between  $\nu_m$  and  $\nu_{m'}$ .

Note that:

- If all the masses are equal (in particular, if they all vanish) there is no oscillation.

- If  $\nu_{\ell_o} = \nu_{m_o}$ ,  $\nu_{\ell_o}$  doesn't oscillate into  $\nu_{\ell'}$  with  $\ell' \neq \ell_o$ .

That is, *oscillation requires neutrinos to have both mass and non-trivial mixing*.

Let us now make some comments on the features of the neutrino oscillations:

**Remarks:**

0 – The oscillatory terms in  $P(\ell \rightarrow \ell', x)$  come from interference between the different mass eigenstates in the neutrino wave function.

1 – If  $x \ll L_{mm'}$ ,  $\nu$  stays in its original flavor.

2 – If  $x$  becomes too large ( $x \gg L_{mm'}$ ), the oscillation pattern will be washed out. This is due to the spread in momentum  $\Delta p$ , in any actual neutrino beam. More precisely, the oscillations will disappear if the neutrino with momentum  $p_\nu$  has a phase shift  $\sim \pi$  compared to the neutrino with momentum  $p'_\nu = p_\nu + \Delta p_\nu/2$ . The average over the oscillatory term will then vanish.

Let  $X$  be the wash-out distance and  $L'_{mm'}$  the oscillation length corresponding to the momentum  $p'_\nu$ . We have:

$$2\pi \frac{X}{L'_{mm'}} = 2\pi \frac{X}{L_{mm'}} - \pi$$

$$L'_{mm'} \simeq L_{mm'} \left( 1 + \frac{\Delta p_\nu}{2p_\nu} \right)$$

therefore: 
$$X \sim \frac{p_\nu}{\Delta p_\nu} L_{mm'} \quad (2.9)$$

The oscillations disappear if the distance  $x$  is bigger than  $X$ . Beyond that point we have:

$$P(\ell \rightarrow \ell', x) = \sum_m U_{\ell m}^2 U_{\ell' m}^2 \neq 0$$

Notice that it is still possible to find an  $\ell'$  neutrino in the  $\ell$  neutrino beam, but the probability for this will no longer vary with distance.

3 – As a conclusion, the oscillation pattern can be observed if the length of the experiment  $x$  is of the order of magnitude of the oscillation length  $L_{mm'}$ .

There are basically two types of neutrino oscillation experiments:

- *Appearance experiments*: these look for  $\nu_\ell \rightarrow \nu_{\ell'}$ , where  $\ell' \neq \ell$ .

- *Disappearance experiments*: these look for a reduction in the flux of neutrinos of the original flavor, or else for an  $x$ -dependence of this flux.

*ii) Neutrino oscillations and the principles of quantum mechanics*

As already stressed, the phenomenon of oscillation between a set of observable states is a basic feature of quantum mechanics. It appears, for example, when a polarized light beam goes through a birefringent crystal or matter made out of chiral molecules. There is also a famous example in particle physics: the neutral kaon system.

We would like now to illustrate that neutrino oscillation is a lovely example of quantum mechanics in action.<sup>12</sup>

Take our typical neutrino oscillation experiment pictured in the figure at the beginning of the previous section. Suppose we see electrons produced in our detector by the neutrinos which started out as muon neutrinos. Suppose, further, that we see the probability  $P(\nu_\mu \rightarrow \nu_e, x)$  oscillating with distance as predicted by formula (2.7).

Now, suppose that we can detect the muons from the pion decays and measure their momenta accurately enough that we know which mass eigenstate  $\nu_m$  was actually emitted in each decay. This is, of course, impossible in practice (for the moment at least), but let us continue with this “Gedankenexperiment”. Now, if we know which  $\nu_m$  actually goes down the beam line in each event, the beam state is no longer a coherent superposition of mass eigenstates: a particular one has been selected. Therefore, the probability of producing a  $\nu_e$  can no longer oscillate with distance, because such oscillation requires interference between different mass eigenstates. The measurements which tell us which  $\nu_m$  is emitted in each pion decay have destroyed the oscillation pattern. It is interesting to understand how they cause this destruction.

The answer is given by the uncertainty principle:<sup>12</sup>

In order to determine the invariant mass  $M_\nu^2$  of the decay neutrino, one has to measure the momenta of both the pion and the muon. The uncertainty principle says that if the momentum of the pion is measured very precisely, the information about its position (the decay point) is lost. More quantitatively, if  $E_\nu$  and  $p_\nu$  are, respectively, the energy and momentum of the neutrino, so that  $M_\nu^2 = E_\nu^2 - p_\nu^2$ , then the error in the  $M_\nu^2$  measurement is given by:

$$\Delta(M_\nu^2) = [(2E_\nu)^2 (\Delta E_\nu)^2 + (2p_\nu)^2 (\Delta p_\nu)^2]^{1/2}$$

Here  $\Delta E_\nu$  and  $\Delta p_\nu$  are the errors in the  $E_\nu$  and  $p_\nu$  measurements, and we are assuming them to be uncorrelated.

Now, it will be possible to determine which mass eigenstate is emitted in each decay only if  $\Delta(M_\nu^2)$  is less than the smallest mass squared difference  $|M_m^2 - M_{m'}^2|$ , which requires that:

$$2p_\nu \Delta p_\nu < |M_m^2 - M_{m'}^2|$$

The uncertainty in the position of the source is then:

$$\Delta x > \frac{2p_\nu}{|M_m^2 - M_{m'}^2|} = \frac{1}{2\pi} L_{mm'}$$

We see that the uncertainty in the position of the source is then bigger than the largest oscillation length, and all oscillations are washed out. If you have identified which mass eigenstate you have produced, you don't know where the source is accurately enough to see oscillations!

If the  $M_m$  are of the order of 10 eV and  $p_\nu$  is bigger than 100 MeV, one will certainly not *accidentally* measure  $p_\nu$  accurately enough to wipe out the oscillation pattern. But one may want to look for neutrino masses of order 100 MeV in pion or kaon decay precisely by finding the mass of the  $\nu_m$  emitted in each meson decay. The identified  $\nu_m$  won't oscillate, of course. Furthermore, for  $M_m = 100$  MeV, and  $M_{m'} \simeq 0$ , the oscillation length would be less than  $10^{-7}$  cm, even if  $p_\nu$  is as big as 1 TeV.

To see oscillation with distance, the source of the neutrino beam must be localized. Thus,  $p_\nu$  cannot be sharp. A wave packet treatment should really be given to account for the spread in momentum. This has been done,<sup>12</sup> and it gives the same result as the simplified calculation we have done here. It also gives a justification of the assumption we made, namely that the different  $\nu_m$  in a beam have a common momentum  $p_\nu$  and different energies  $E_m$ , rather than, say, a common energy  $E$ , and different momenta  $p_m$ . More details on these problems can be found in the paper referred to above.

iii) *Two-neutrino oscillation*

Suppose that only two neutrino flavor eigenstates,  $\nu_{t_a}$  and  $\nu_{t_b}$ , and two mass eigenstates,  $\nu_1$  and  $\nu_2$ , participate appreciably in the neutrino mixing. Then:

$$\begin{aligned}\nu_{t_a} &= \nu_1 \cos\theta + \nu_2 \sin\theta \\ \nu_{t_b} &= -\nu_1 \sin\theta + \nu_2 \cos\theta\end{aligned}$$

so that the mixing matrix  $U$  is  $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ , a  $2 \times 2$  rotation matrix.

There is now only one oscillation length, namely  $L = \frac{4\pi p_\nu}{|M_2^2 - M_1^2|} = \frac{4\pi p_\nu}{\delta M^2}$ .

From Eq. (2.7), the probability  $P(\nu_{t_a} \rightarrow \nu_{t_b}, x)$  that would be studied in an appearance experiment is given by:

$$P(\nu_{t_a} \rightarrow \nu_{t_b}, x) = \sin^2 2\theta \sin^2 \pi \frac{x}{L} = \sin^2 2\theta \sin^2 \delta_{12} \quad (2.10)$$

$$\delta_{12} = \pi \frac{x}{L} = \frac{\delta M^2 x}{4p_\nu}$$

By conservation of probability, the probability  $P(\nu_{t_a} \rightarrow \nu_{t_a}, x)$  that would be studied in a disappearance experiment is then given by:

$$P(\nu_{t_a} \rightarrow \nu_{t_a}, x) = 1 - \sin^2 2\theta \sin^2 \pi \frac{x}{L} \quad (2.11)$$

We would like to use real units instead of the natural units where  $\hbar = c = 1$ . The apparent dimension of the dimensionless  $\delta_{12}$  is [length]·[energy], so

$$\delta_{12} = \frac{1}{4} \frac{1}{\hbar c} \frac{\delta M^2 x}{p_\nu} = \frac{1}{4} \frac{10^{18} \cdot 10^{-18}}{0.197} \frac{\frac{\delta M^2}{1\text{eV}^2} \cdot \frac{x}{1\text{km}}}{\frac{p_\nu}{1\text{GeV}}}$$

or

$$\delta_{12} = 1.27 \frac{\frac{\delta M^2}{1\text{eV}^2} \cdot \frac{x}{1\text{km}}}{\frac{p_\nu}{1\text{GeV}}} \quad (2.12)$$

This very useful formula gives the sensitivity of an experiment. An experiment usually aims to be sensitive to small  $\delta M^2$ . The ratio  $x/p$ , therefore has to be as big as possible. The best we can do is to use low energy neutrinos which travel a very long distance. For example, the IMB experiment has studied neutrinos created in the atmosphere by cosmic rays. It compares the flux of neutrinos born just above the detector with that of neutrinos which are born on the opposite side of the earth, and which must then traverse the whole earth ( $x \sim 10^4$  km) to reach the detector (see Figure 3). This comparison gives a sensitivity to  $\delta M^2 \sim 10^{-4} \text{eV}^2$  for reasonably large mixing angles.<sup>13</sup>

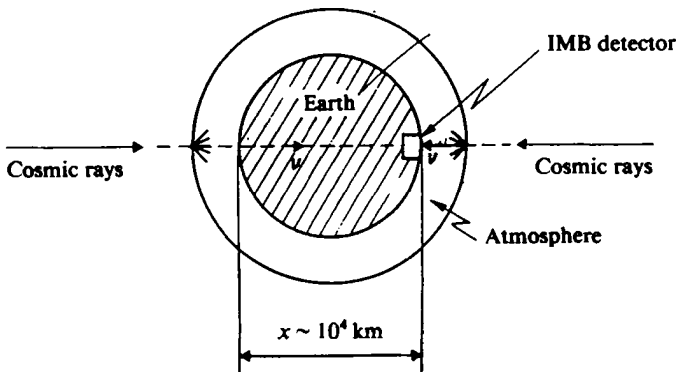


Fig. 3. An experiment to observe the oscillation of atmospheric neutrinos.

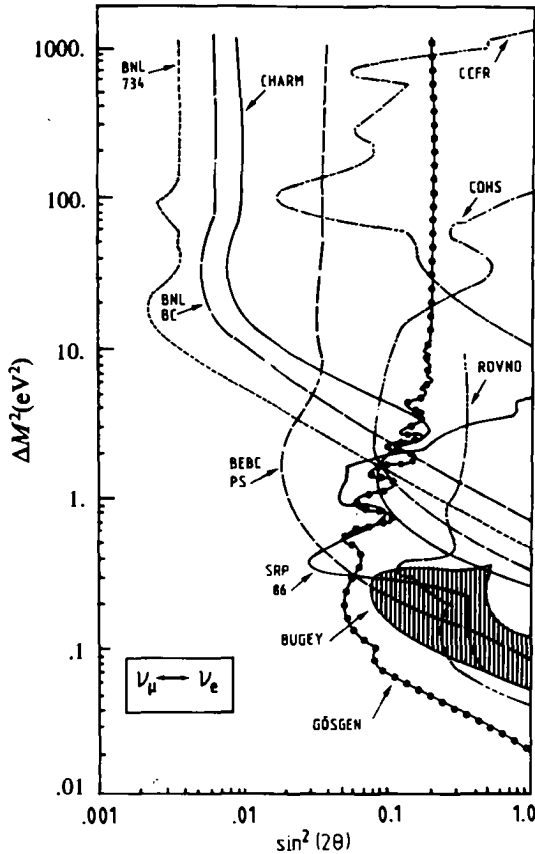


Fig. 4a. Limits in the plane  $(\delta M^2, \sin^2 2\theta)$  for the channel  $\nu_\mu \leftrightarrow \nu_e$ . (The shaded area corresponds to the region allowed by the BUGEY experiment.)

Almost all the experiments performed up to now have given negative results, and limits in the plane  $\delta M^2$  versus  $\sin^2 2\theta$  have been extracted. (Figure 4 a, b, c show those limits for three different channels, using data from both appearance and disappearance experiments.<sup>14)</sup>

#### iv) Special oscillations

Apart from the oscillations we have already considered, there is also the possibility that a known neutrino, like  $\nu_\mu$ , could oscillate into a so-far

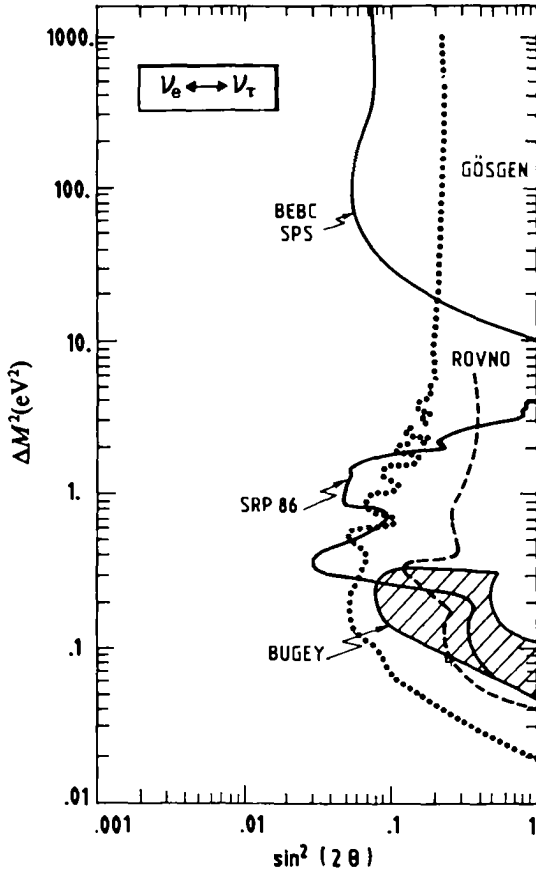


Fig. 4b. Limits in the plane  $(\delta M^2, \sin^2 2\theta)$  for the channel  $\nu_e \leftrightarrow \nu_\tau$ . (The shaded area corresponds to the region allowed by the BUGEY experiment.)

unknown neutral lepton. For example, suppose there is a fourth lepton generation, containing a heavy charged lepton  $\Phi$ , and an associated light neutrino  $\nu_\Phi$ . If a  $\nu_\mu$  whose energy is below the  $\Phi$  production threshold oscillates into a  $\nu_\Phi$ , the  $\nu_\Phi$  will be unable to participate in charged-current interactions. It will be able to participate in neutral-current interactions, but if the detector observes only charged-current interactions, it will appear as if the  $\nu_\mu$  has turned into a sterile particle.

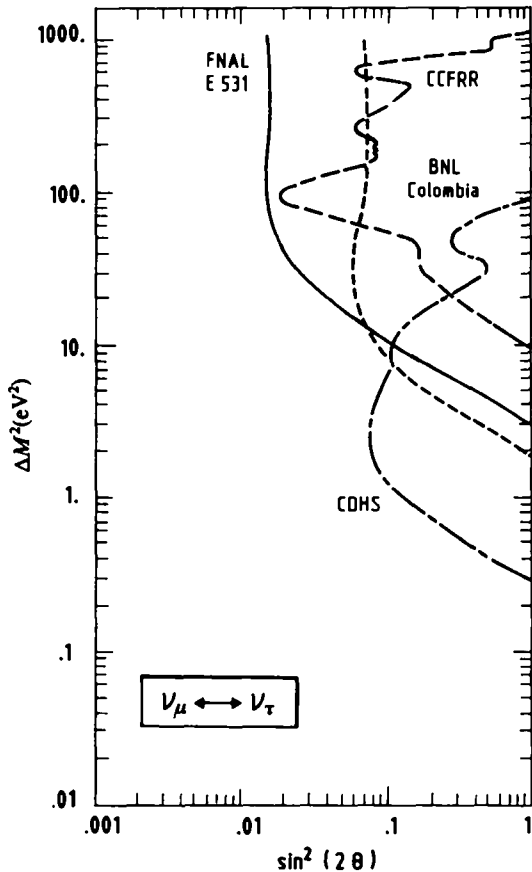
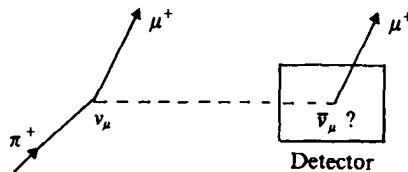


Fig. 4c. Limits in the plane ( $\delta M^2$ ,  $\sin^2 2\theta$ ) for the channel  $\nu_\mu \leftrightarrow \nu_\tau$ .

In addition, a known particle such as a  $\nu_\mu$  may oscillate into a truly sterile neutrino  $\nu_s$  which does not participate in either the ordinary charged-current or neutral-current interactions at any energy. If this occurs, some of the flux in a  $\nu_\mu$  beam will simply disappear, without reappearing as  $\nu_e$  or  $\nu_\tau$  flux.

As these two examples show, a disappearance oscillation experiment does not necessarily measure the same thing as an appearance experiment.

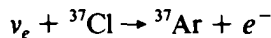
If neutrinos are Majorana particles, it is also possible for a  $\nu_\mu$ , interacting with matter, to produce a  $\mu^+$  or an  $e^+$ , rather than a  $\mu^-$ . Although this effect varies with distance, it can actually occur as soon as the  $\nu_\mu$  is created. The point is that there is a small probability, of order  $(M_m/E_m^*)^2$ , for a given  $\nu_m$  component of the  $\nu_\mu$  to be produced with positive helicity. Here  $E_m^*$  is the energy of the  $\nu_m$  in the rest frame of the particle (a pion, say) whose decay produces the neutrino. Now, in the Majorana case, a positive-helicity “neutrino” is what we normally call an antineutrino, and it will produce a positively charged lepton when interacting with matter. This leads to phenomena such as the one pictured below. The trouble is that  $(M_m/E_m^*)^2$  is likely to be extremely tiny, so that the effect cannot be observed.



Up to now, we have discussed oscillation in vacuum. However, Mikheyev and Smirnov,<sup>15</sup> following earlier work of Wolfenstein,<sup>16</sup> have found recently a very exciting phenomenon which can occur when neutrinos travel through a dense material, and we will say a few things about it.

### b) Neutrino flavor transitions in matter

Transitions from one neutrino flavor to another may have a very important connection to the famous solar neutrino mystery. Let us recall what this mystery is: the flux of electron neutrinos from the sun has been measured experimentally using the reaction:



These neutrinos are coming from nuclear reactions occurring in the core of the sun, and the flux can be computed using the standard model of

this star. The trouble is that the measured flux is a factor 4 too small compared to the calculation.

One idea to solve this discrepancy is to introduce neutrino masses and mixing between different neutrino species. Oscillation can then occur, and the  $\nu_e$  flux on the earth is reduced. The distance between the source and the detector is very large, presumably a large multiple of the oscillation length, so that (see Eq. (2.9) and subsequent discussion) the reduction is given by:

$$P_{ee} = \sum_m |U_{em}|^4$$

If we accept the likely hypothesis that there is a mass eigenstate very “close” to  $\nu_e$ , we find:

$$P_{ee} \sim 1$$

This means that the neutrino oscillation in vacuum is unlikely to be the explanation for the factor-of-four reduction of the solar  $\nu_e$  flux. If we want to keep the phenomenon of neutrino flavor transition as an explanation, it seems that we have to assume that the mixing angles are very large. However, this is actually not necessary. If the neutrinos propagate through matter, the flavor transition pattern can be changed and one can get a very large reduction of the flux even with a small mixing angle. How does it work?

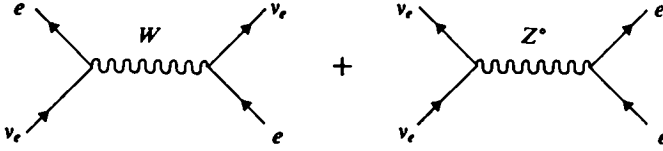
Let us take for example a two neutrino oscillation  $\nu_e - \nu_\mu$ . In vacuum we have:

$$\begin{aligned} \nu_e &= \nu_1 \cos\theta_v + \nu_2 \sin\theta_v \\ \nu_\mu &= -\nu_1 \sin\theta_v + \nu_2 \cos\theta_v \end{aligned}$$

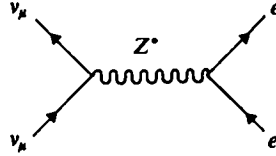
where  $\nu_1$  and  $\nu_2$  are the mass eigenstates, which propagate in vacuum without mixing, and  $\theta_v$  is the mixing angle in vacuum.

If these neutrinos travel through the sun, they will interact with electrons and the propagation of  $\nu_1$  and  $\nu_2$  will be modified by the fact that  $\nu_e$  and  $\nu_\mu$  have different interactions with electrons.

A  $\nu_e$  interacts with an electron via both  $W$  and  $Z^\circ$  exchange,



while a  $\nu_\mu$  (or a  $\nu_\tau$ ) interacts only via  $Z^0$  exchange,



As a result of this difference, the eigenstates which propagate in solar matter without mixing (the “solar eigenstates”) differ from those which propagate without mixing in vacuum (the mass eigenstates). The mixing angle  $\theta_M$  in the solar matter (the rotation angle between  $\nu_e - \nu_\mu$  and the solar eigenstates) is related to the mixing angle  $\theta_v$  in vacuum (the rotation angle between  $\nu_e - \nu_\mu$  and the mass eigenstates) by:

$$\sin^2 2\theta_M = \frac{\sin^2 2\theta_v}{\left(\cos 2\theta_v - \frac{L_v}{L_o}\right)^2 + \sin^2 2\theta_v} \quad (2.13)$$

Here  $L_v$  is the oscillation length in vacuum, and  $L_o$ , a length characteristic of the motion of the neutrinos through the solar matter, is defined by:

$$L_o = \frac{2\pi}{\sqrt{2} G_F N_e}$$

$G_F$  is the Fermi constant

$N_e$  is the number of electrons per unit volume in the sun.

Mikheyev and Smirnov have pointed out that Eq. (2.13) has a resonant character. Suppose that on their way from the solar core, where they are

produced, to the outer edge of the sun, the solar neutrinos encounter a region where the electron density  $N_e$  is such that

$$\frac{L_v}{L_o} = \cos 2\theta_v$$

Then, according to Eq. (2.13), the mixing angle  $\theta_M$  will be maximal,

$$\sin^2 2\theta_M = 1$$

even if the vacuum mixing angle  $\theta_v$  is very small. Thus, very appreciable  $\nu_e \rightarrow \nu_\mu$  conversion may occur in the sun, even for small  $\theta_v$ . This Mikheyev-Smirnov-Wolfenstein conversion may be the explanation of the solar neutrino mystery. Interestingly, the values of  $\delta M^2$  which can lead to the  $\nu_e$  flux reduction observed in the  $^{37}\text{Cl}$  solar neutrino experiment lie in the range  $10^{-4} \text{ eV}^2$  to  $10^{-8} \text{ eV}^2$ .<sup>17</sup> Thus, future study of solar neutrinos may be a means to probe neutrino masses well below 1 eV.

## EXERCISES

1 – We have derived the probability of neutrino oscillation,  $P(\ell \rightarrow \ell', x)$ , by assuming that a neutrino born as  $\nu_\ell = \sum_m U_{\ell m} \nu_m$  has a well-defined momentum  $p_\nu$ , so that its various mass eigenstate components  $\nu_m$  have different energies  $E_m = \sqrt{p_\nu^2 + M_m^2}$ . Assume, instead, that this neutrino has a well-defined *energy*  $E_\nu$ , so that its various mass eigenstate components have different *momenta*  $p_m = \sqrt{E_\nu^2 - M_m^2}$ . Show that this alternative assumption leads to the same wave function  $\psi(x, x)$ , Eq. (2.5), as before, except that  $p_\nu$  is replaced by  $E_\nu$ . Thus, this assumption leads to the same oscillation pattern as before.

2 – For the  $\nu_e$  flux from the sun to be reduced appreciably by neutrino oscillation in vacuum, neutrino mixing must be large. Assuming that the earth is many oscillation lengths from the sun, and that CP is conserved,

vacuum oscillation reduces the  $\nu_e$  flux by the factor:

$$P_{ee} = \sum_{m=1}^N U_{em}^4$$

Show that, if there are  $N$  different types of neutrinos, the minimum possible value of  $P_{ee}$  is  $1/N$ . Show further that if  $N$  is odd (three, for example), then the orthogonality of  $U$  makes it impossible for the flux of more than one neutrino flavor to be reduced to the minimum value  $1/N$ .

3 – Let  $N_e(r)$  be the density of electrons in the sun as a function of the radius  $r$ , and let  $\Delta r$  be the thickness of a critical region,  $r_{\text{crit}} - \frac{\Delta r}{2} <$

$r < r_{\text{crit}} + \frac{\Delta r}{2}$ , in which  $N_e(r)$  is such that for neutrinos of a certain energy the solar mixing parameter  $\sin^2 2\theta_M$  given by Eq. (2.13) exceeds, say,  $1/2$ . If we are to have a very high degree of  $\nu_e \rightarrow \nu_\mu$  conversion in this region by the Mikheyev-Smirnov-Wolfenstein mechanism, then  $\Delta r$  must be large enough to give the neutrinos ample time to convert before they reach a radius where  $\sin^2 2\theta_M$  has become small again. Thus, we must have  $\Delta r > L_M$ , where  $L_M$  is the oscillation length of the neutrinos in the material of the critical region. Using the fact that  $L_M = L_v/\sin 2\theta_v$ ,<sup>16</sup> show that the condition  $\Delta r > L_M$  requires that:

$$\tan^2 2\theta_v \gtrsim \left[ -\frac{1}{G_F N_e^2} \frac{dN_e}{dr} \right]_{r=r_{\text{crit}}}$$

Fortunately, this “adiabaticity constraint” on  $\theta_v$  can be satisfied by quite small values.

## PHYSICS OF MASSIVE NEUTRAL LEPTONS

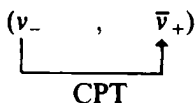
Having spent some time discussing the phenomenology, we would like now to discuss more extensively the physics of massive neutral leptons and in particular the physics of objects that are their own antiparticles, which is not very familiar. We shall study the properties of Majorana particles quite in detail. First we shall see that we can deduce a lot of interesting properties without using any formalism. Then, we shall discuss the formalism and write down the quantum field of a Majorana particle.

### 1 – PHYSICS WITHOUT THE FORMALISM OF FIELD THEORY

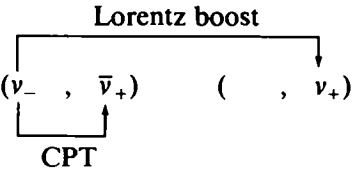
#### a) An approach to Majorana and Dirac particles

Here we shall make use only of basic quantum mechanics and discrete symmetries. We shall take our time to review  $C$ ,  $P$  and  $T$  operators and then define precisely what we call a Majorana neutrino.

Suppose there exists a massive neutrino with negative helicity  $\nu_-$ . However, our theory is CPT invariant: this means that there exists also the CPT mirror image of  $\nu_-$ , an antineutrino with positive helicity:  $\bar{\nu}_+$ .

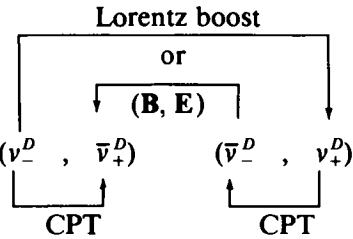


But our neutrino  $\nu_-$  has a mass: its velocity is smaller than the speed of light so an observer can move faster. Then, in the frame of this observer, the neutrino is running the other way around but it is still spinning the same way. In other words, we have converted with a Lorentz boost our neutrino with negative helicity into a neutrino with positive helicity  $\nu_+$ .



Now the question is: are these two states with positive helicity  $\nu_+$  and  $\bar{\nu}_+$  the same? (That is, do they interact in the same way?).

- If we make the assumption that  $\nu_+$  is *not* the same particle as  $\bar{\nu}_+$ , then  $\nu_+$  has its own CPT mirror image  $\bar{\nu}_-$ . This new state can be connected to  $\bar{\nu}_+$  by a Lorentz transformation.



We have got four states with the same mass. *This set of states is called a Dirac neutrino  $\nu^D$ .*

In general, a Dirac neutrino will have a magnetic dipole moment and an electric dipole moment. The two states  $(\nu_-^D, \bar{\nu}_+^D)$  can then be converted into their partners  $(\nu_+^D, \bar{\nu}_-^D)$  either by a Lorentz transformation or by the action of an external electromagnetic field  $\mathbf{B}, \mathbf{E}$ .

- If  $\nu_+$  is identical to  $\bar{\nu}_+$ , we are left with only two states with the same mass.