

Mendel Sachs

PHYSICS OF THE UNIVERSE

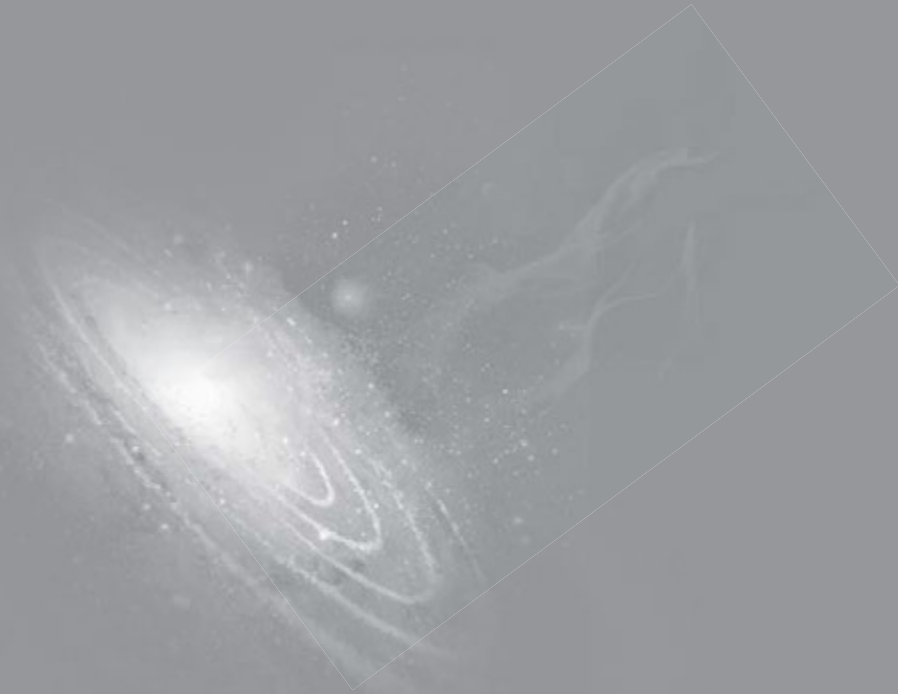
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Mendel Sachs

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Dedicated to

Albert Einstein

Physicist — Philosopher — Humanitarian

And his Quest for Fundamental Truths of the Universe

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Preface

This monograph presents a fresh look at the problems of the physics of the universe — the subject of cosmology — compared with other contemporary cosmological theories. Its main purpose is to fully exploit the most fundamental expression of Einstein's theory of general relativity as the mathematical description (language) and the explanation of the physical behavior of the material universe. Contemporary quantum theory, the 'standard model' of elementary particle physics and thermodynamics, play no fundamental role in this analysis.

Physics is an empirical science. That is, its claims to understand the material behavior of the objects of the universe, with a logically consistent theory, must be backed by agreement between the theoretical predictions and the empirical facts. Newton's theory of universal gravitation was believed to be true for the three centuries that preceded Einstein's discovery of general relativity. But in the context of cosmology, Newton had no empirical backing for anything beyond our solar system — an infinitesimal portion of the universe! In the 20th century new experimental facts came to light on the problem of cosmology that did not fit Newton's theory. Primary was Edwin Hubble's discovery of the expansion of the universe. More recently, high-resolution instrumentation for observing the universe, such as the Hubble telescope, revealed observations unknown to Newton or his contemporaries, and his followers during the next three centuries. It has led to the revelation of a universe composed of an infinitude of galaxies, in addition to

our own galaxy, the Milky Way, the spiral structures and rotations of most of the galaxies, the clustering of the galaxies, the background radiation field of the universe, and to exotic stars, such as the quasars and pulsars.

In the early decades of the 20th century, Einstein's theory of general relativity reproduced all the successful results of Newton's theory of universal gravitation, as well as new empirical predictions of gravitational effects not at all predicted by the classical theory. Thus Einstein's general relativity superseded Newton's theory as an explanation of the phenomenon of gravity. Einstein's theory of gravitation was entirely different to that of Newton, conceptually and mathematically. That is to say, Newton's theory of universal gravitation *is not included* in Einstein's theory of general relativity. Rather, Newton's formalism is not more than a mathematical approximation for that of Einstein's theory of general relativity.

Newton's theory is based on a model of the universe as an open system, on atomism and on the concept of action-at-a-distance. Newton himself was dissatisfied with this as an *explanation* of gravitation, but he accepted it as a useful *description*, for his day. In his theory of gravitation, space and time are separate measures in the description of the gravitational force. In contrast, Einstein's theory entails the fusion of the space and time measures in the language of the laws of nature. In this generalization of the language, the space and time measures cannot be objectively separated. They form a fused spacetime, wherein a spatial (or temporal) measure in the language of a law of nature in one frame of reference is a mixture of a spatial and a temporal measure in a different reference frame where the same law is to be compared. Here is the idea of the universe as a closed system, where forces between matter components propagate at a finite speed. These matter components of the universe, in turn, are not separate, singular entities. Rather, they are the distinguishable correlated modes of the closed, continuous matter of the universe. Instead of atomism, this view is then based on the continuous, holistic field concept. (It is compatible with Mach's interpretation of the inertial mass of matter, named by Einstein, 'the Mach principle'.) *This is a genuine paradigm change from the atomistic*

view that has dominated physics for the many centuries since ancient Greece. An exception in the history of science was Michael Faraday's support of the continuous field concept, in the 19th century.

One of the important predictions of the Einstein theory was that in the description of the laws of nature, because the space and time measures are not separate, but are rather a fused spacetime measure, there is the implication that the objects of the universe are not in stationary orbits about other objects, as Newton's theory would hold. Instead, Einstein's theory is in agreement with the Hubble observation that the matter components of the universe are not in stationary orbits, rather they are moving away from all other material components of the universe, i.e. that *the universe is expanding*. Or in a different phase of an oscillating universe cosmology, they are moving toward all other matter components in a contraction phase of the universe.

Are there any deficiencies in Einstein's theory of general relativity, based on its own premises? Yes, there are. It was Einstein who said that to fully exploit the theory of general relativity, not only must one study its geometrical basis, but also its algebraic basis. The latter refers to the most general expression of the symmetry group that underlies this theory of matter. In my studies I have found that this is a *continuous group (indeed, it is a continuous group of analytic transformations — thus a 'Lie Group'. I have called it the 'Einstein Group')*. But the group of transformations that leave Einstein's tensor field equations covariant (i.e. frame-independent) are not only continuous, as they must be, but they are also reflection symmetric, which is not required. Thus Einstein's tensor equations are *not* the most general form for his theory of general relativity.

What I have shown in this monograph is that when the reflection symmetry elements of the underlying symmetry group are removed, Einstein's (reducible) tensor field equations factorize to an (irreducible) quaternion field equation. The 10-component, symmetric metric tensor $g^{\mu\nu}(x)$ of Einstein's original formulation of general relativity is then replaced with the 16-component quaternion metric field $q^{\mu}(x)$. This is covariant as a four-vector under the continuous transformations that characterize the Einstein group,

but it is not covariant under the discrete reflection elements that characterize the symmetry of Einstein's tensor field equations. The quaternion field $q^\mu(x)$ is a four-vector in which each of its four components is quaternion-valued. Thus this field has $4 \times 4 = 16$ independent components. The new quaternion metrical field equations are then 16 in number. I have shown in Chapter 3 (pp. 27–36) that this leads to a unified field theory — by iteration it is seen that ten of the components correspond exactly with Einstein's tensor formulation, thus explaining gravity. The remaining six equations give the Maxwell field equations in the three electric and three magnetic field variables, thus explaining electromagnetism. **This is the unified field theory that was sought by Einstein and Schrödinger.**

In Chapter 4 (pp. 37–50) it is shown that under the appropriate conditions, the new quaternion formulation of general relativity leads to: 1) The (empirically correct) Hubble law, as a non-covariant approximation for the dynamics of an expanding universe and 2) the spiral structure of an oscillating universe, alternating between expansion and contraction, in the never-ending cycles of the universe. Thus, with this view, there is no absolute (singular) 'beginning' of the universe. An absolute temporal beginning of the universe is indeed ruled out as incompatible with the relativity of the time measure in Einstein's theory of relativity. Further, the spacetime is necessarily curved, *everywhere*. The flat spacetime is only an ideal, unreachable limit that simply characterizes the universe as a vacuum *everywhere*.

In Chapter 5 (pp. 51–59) it is shown that a viable candidate for the dark matter of the universe is a dense sea of particle-antiparticle pairs (electron-positron and proton-antiproton) in a particular (*derived*) bound state of null energy, momentum and angular momentum — it is the true ground state of the pair; the sea of such pairs in this state is likened in other theories to the 'vacuum state'.

Many of these results are derived in more detail in previous publications of this author. The main purpose of this presentation is to focus discussion primarily on the problems of cosmology and to indicate their resolutions in a maximally explanatory way.

There are some brief, concluding remarks in the Chapters 6 and 7 on the subject of black holes and pulsars, as well as philosophical considerations.

Summing up, the language of the physics of the universe — the language of cosmology — is presented here in terms of Einstein's theory of general relativity, expressed in its *irreducible form* by removing the reflection symmetry elements from the underlying group (thus yielding the irreducible symmetry group — the 'Einstein group'). This leads to a unified field theory in terms of the generalization of the symmetric tensor representation (10 independent field equations) to the quaternion representation (16 independent field equations). This unification explains, in fundamental terms, gravitation and electromagnetism in a single formalism.

The (irreducible) quaternion formalism, as the language for cosmology, then leads to a new view of the universe. The dynamics of the universe is seen in terms of a cyclical, oscillating model, alternating between expansion and contraction. Further, this model is not in terms of an isotropic and homogeneous matter distribution of the universe, as the present-day cosmological theories contend. Rather, it is in terms of a spiral, rotating material universe, in a curved spacetime. The Hubble law is predicted here as an approximation for the dynamics of any expansion phase of the oscillating universe. With the view of 'dark matter' developed here, a prediction of this model is the empirically observed separation between matter and antimatter in the universe, occurring during the (non-singular!) 'big bang' that initiates any expansion phase. In contrast with present-day models that evoke concepts of particle physics — the quantum theory, the standard model, the string theories — these play no role in this dynamical representation of the physics of the universe. It is explained here strictly in terms of the theory of general relativity and its irreducible expression, based on the underlying *principle of covariance*.

Mendel Sachs
Buffalo, New York
July, 2009
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My research program on cosmology, based on general relativity theory and unified field theory, was inspired by the writings of Albert Einstein and Erwin Schrödinger. I gained also from the 19th-century writings of William Rowan Hamilton and his discovery of the quaternion algebra. I had the good fortune to spend some time in discussion on these subjects with Cornelius Lanczos, at the Dublin Institute for Advanced Studies. Last, but not least, I thank my son, Robert, for persuading me to write this book.

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Physics of the Universe

Introduction

Physics is the science of inanimate matter. Cosmology is the part of this science that deals with the universe as a whole. It is the oldest and the youngest branch of physics. It is the oldest because the heavens were studied in ancient times, in Greece and in Asia, and other parts of the world. It is the youngest because it has been re-invigorated in recent times due to observations with new, high-resolution astronomical instrumentation (such as the Hubble telescope) and theoretical analyses in the context of current thinking in particle physics and relativistic dynamics. Voluminous works have been written on the order of the night sky. (*The Greek word, 'Cosmology', means 'order' (logos) of the cosmos.*)¹

Astronomical laboratories have been constructed since the ancient times to study this order. Examples include the Stonehenge monument, built by the ancient Britons thousands of years ago, and similar ancient astronomical viewing sites in India, China, Australia, Peru, Mexico and from other cultures in the different corners of the world, designed by the ancient and aboriginal peoples to see the star formations and their locations, the locations of the sun and the moon, at the different times of the year. In these ancient viewings, there was no magnification.

Galileo, in the 16th century, was the first astronomer to use magnification, utilizing the telescope — a series of lenses that he

contrived to view the heavens.² Focusing mainly on our solar system, he saw the moons of Jupiter, the sun spots, the landscape of the moon, and he verified the conclusion of Copernicus that *the earth moves!* However, Galileo went further than Copernicus, who theorized that the sun is at the absolute center of the universe, and that the earth orbits about it, along with the other planets of the solar system. Galileo said there is no absolute center of the universe and that ‘motion’ *per se* is not an objective quality of matter.³ Rather, it is a subjective element in its description. Thus, in Galileo’s view, it is just as true to say, *from the earth’s perspective*, (i.e. its frame of reference) that the sun rotates about the earth, as it is to say that, *from the sun’s perspective*, the earth rotates about the sun. Indeed, it was Galileo’s view that the laws of nature underlying the binding of the earth to the sun (and vice versa) are independent of the perspectives taken from which the observations of their binding ensues. This is called *Galileo’s principle of relativity*.⁴ It is an important precursor for ‘*Einstein’s principle of relativity*’ that underlies his theory of relativity.

In the generation that followed Galileo, Newton (who was born the same year as Galileo died, 1642) discovered the laws of motion of things (discrete masses) and the law of universal gravitation. He also perfected the means of viewing the night sky with a new, higher-resolution type of telescope — a ‘reflecting telescope’ — that was a collection of mirrors rather than lenses, much smaller in dimension than Galileo’s lens-type telescope. Newton also did research in his attempt to understand optical phenomena. His view was a mechanistic one, where light is assumed to be a collection of particles. (*Newton’s contemporaries, Hooke and Huygens, believed in a continuous wave theory of light. It was later verified that Newton’s corpuscular model of light was wrong, that light is really a continuous wave phenomenon, whose underlying basis is electromagnetic radiation.*)⁵

In the 18th century, William Herschel discovered that the ‘Milky Way’ is not the entire universe, as Galileo believed. Herschel found that our galaxy, ‘Milky Way’, has a neighboring galaxy of stars, called ‘Andromeda’.⁶ This was discovered later on to be a member of a

binary system with 'Milky Way'. Later in the history of astronomy, it was found that the universe consists of an indefinitely large number of galaxies, each containing an indefinitely large number of stars. The galaxies are not distributed in the universe homogeneously and isotropically. Rather, they cluster in certain regions and are absent in others.

With the new high-resolution instrumentation, it was found in the 20th century that most of the galaxies are pancake-shaped, with spiral arms, where the highest density of stars is toward their centers. Our sun is an average-sized star residing in one of the spiral arms of the 'Milky Way'. We also know that the galaxies rotate about an axis that is perpendicular to their planes. Some of the galaxies that do not have spiral arms have the (egg-like) shapes of ellipsoids. It is possible these are in an evolutionary stage, later to develop spiral arms. It was found that the rotations of the galaxies cannot be attributed to the Newtonian gravitational pull of the neighboring galaxies. They would not have the sufficient magnitude. It has been proposed that there is unseen *dark matter* in the regions of the galaxies (and throughout the universe) that is responsible for their rotations. The details of dark matter will be discussed in Chapter 5 (pp. 51–59).

Newton's equations of motion and his theory of universal gravitation imply all of the heavenly bodies are in stationary orbits about other bodies, just as the planets of our solar system, *from the perspective of the sun*, are in stationary orbits about the sun. This is because of the separation of the (relative) functional dependence on spatial coordinates from the functional dependence on the (absolute) time coordinates in Newton's equations of motion. The equation of motion in the spatial coordinates, in Newton's theory, predicts the elliptical orbits and constant angular momenta of a rotating body, such as the earth, relative to its center of rotation, the sun, whose center of mass resides at one of the elliptical foci. This confirms Kepler's discovery, a generation before Newton, that the orbit of Mars is elliptical about the sun, which is at one of the elliptical foci, and his generalization of this observation to a law of orbital motion of all the planets.

Is Newton's Theory an Explanation of Gravity?

One feature of Newton's theory that puzzled him was 'action at a distance'. That is, a body spontaneously acts on another body, irrespective of the magnitude of their spatial separation. He said the following about this concept: *'Action at a distance through a vacuum, without mediation of anything else by and through which their action and force may be conveyed from one to another, is to me so great an absurdity, that I believe no man who has in philosophical matters competent faculty of thinking, can ever fall into it'*. He said this in a letter to R. Bentley III, in 1693. [See A. Janiak (ed.); *Newton: Philosophical Writings* (Cambridge, 2004), p. 102. Also see M.B. Hesse, *Forces and Fields* (Nelson, 1961), p. 152]. But he excused his use of the concept by saying: 'because it works! I do not form hypotheses'. Yet, from a study of his writings, it is clear that Newton did indeed form hypotheses in physics! Because of his unhappiness with 'action at a distance', I believe Newton did not yet think that he had *explained* gravitation, though he *described* it adequately for his time.

It was not until the fruition of Einstein's theory of general relativity, three centuries later, that gravitation was explained satisfactorily. Two features that differed in Newton's versus Einstein's theories of gravitation were: (1) atomism versus the continuous field concept and (2) action at a distance versus the propagation of forces at a finite speed. A further difference was that in Newton's theory, the space parameters and the time parameters are separate from each other in the sense that the time measure is absolute — it does not change in the expression of a law of nature from one reference frame to another, but a spatial measure in one reference frame is transformed to different spatial measures in other reference frames where the law of nature is expressed. But in the theory of relativity, a spatial (or a time) measure is transformed to a mixture of a spatial and a time measure in other reference frames where the laws of nature are expressed. The time measure then becomes relative to the reference frame in which the law of nature is expressed. This is in contrast to Newton's classical theory, where the time measure is absolute, i.e. frame-independent. Thus, in relativity theory, space and time

become *spacetime*. Such a language for the laws of nature then does not predict stationary orbits for the heavenly bodies.

The Expanding Universe

The lack of stationary orbits of the heavenly bodies of the universe was verified by Hubble in his discovery of the expanding universe.⁷ This observation was in agreement with the prediction of no stationary orbits of matter of the universe, according to Einstein's theory. In our local domain of the solar system, at first glance there seems to be the stationary orbits of the planets relative to the sun, but Einstein's theory showed that these are really not stationary either, as exemplified in the observation of the perihelion precession of Mercury's orbit (i.e. the planet Mercury does not return to the same place relative to the sun, after each cycle of its orbital motion). Hubble's data revealed the fact that the galaxies of the universe are moving away from each other at an accelerating rate, in accordance with the Hubble law: $v = Hr$, where v is the speed of one galaxy relative to another and r is their mutual separation. This cosmological dynamics is referred to as 'the expanding universe'.

This dynamics does not mean to imply that the matter of the universe, as a whole, is expanding into empty space! For there is no empty space as a 'thing in itself'. The universe is, by definition, all there is. What is meant, physically, by the 'expansion' of the universe is that from any observer's view, the density of matter, anywhere in the universe, is decreasing with respect to his or her time measure.

The way that Hubble detected this expansion was to measure the Doppler Effect of the radiation emitted by the galaxies. If a source of radiation emission is moving away from an absorber of this radiation, its absorbed frequency components will shift toward the longer wavelength end of the spectrum — this is a 'red shift' for the visible spectrum. Thus, Hubble saw the spectrum from a more distant galaxy red shifted from a similar spectrum of a closer galaxy. This would mean that the density of matter in the observed stellar domain of the universe is decreasing in time, according to any particular observer's measurements.

If the universe is indeed monotonically expanding in this way, then if one should extrapolate to the past, the universe was increasingly dense at the earlier times. The extrapolation would then reach a limit, where the matter of the universe was maximally dense and unstable. At that point in time, according to any observer's view, the universe would have exploded, starting off the presently observed expansion. This event of the universe is commonly referred to as the 'big bang'.⁸

The question then arises: how did the matter of the universe get into this state of maximum density and instability in the first place? This is a scientific question and requires a scientific answer. To answer this question theologically by saying that this point in time was when God created the universe is a nonsequitur. It is based on religious truth, founded on irrefutable faith. That is not to say religious truth is false. Rather, it is in a different context to scientific truth. Scientific truth is in principle refutable and based on scientifically testable empirical facts and logical consistency of the basis of this theory. This difference will be discussed further in Chapter 7 (pp. 64–73).

The Oscillating Universe Cosmology

The only *scientific* answer to this question that I see is this: before the big bang event and the subsequent expansion phase of the universe, where the dominant gravitational force between galaxies was repulsive, the universe was imploding with a dominant attractive gravitational force between the matter components, contracting as a whole toward ever-increasing density. An inflection point was then reached, where the dominant attractive forces changed to dominant repulsive forces, and the contraction changed to an expansion. Eventually, when the density of the matter of the presently expanding universe will become sufficiently rarefied, another inflection point will be reached, where the dominant repulsive forces between the matter components of the expanding universe will change to a dominant attractive force. Once again, the expansion will then change to a contraction and the matter of the universe will implode

until the next ‘big bang’, initiating the next expansion phase. The dynamics of the universe as a whole continue in cyclic fashion between expansion and contraction. This is compatible with the meaning of time in relativity theory as a non-absolute measure. That is to say, this model in cosmology rules out the concept of an absolute temporal *beginning* of the universe, from a scientific stand.

The oscillating universe cosmology follows in physics from the theory of general relativity. The terms that play the role of force in this theory depend on the geometrical functions of the curved spacetime, called ‘affine connection’.⁹ They are not positive-definite. Thus, under one set of physical conditions, when matter is sufficiently dense and the relative speeds of matter are close to the speed of light, the gravitational forces are repulsive, predicting that matter moves away from other matter. But when the matter becomes sufficiently rarefied in the expansion phase, the dominant repulsive forces between matter become dominantly attractive, leading to the contraction of the matter of the universe. (In the latter phase of the oscillating universe, the *redshift* of radiation observed by Hubble, implying an expansion would change to a *blueshift* — a shift of the visible spectrum toward the shorter wavelengths, as observed when the source of the radiation is approaching the absorber of this radiation, rather than receding from the absorber. This would imply a contracting universe.)

Thus, with this scenario, the universe is continually expanding and contracting — there were continual ‘big bangs’ in the indefinite past and will continue in the indefinite future. The time of the last ‘big bang’, estimated (from the Hubble law) to be about 15 billion years ago, was only the beginning of this particular cycle of the oscillating universe.

The Theory of General Relativity

The underlying physical dynamics of the universe as a whole is given by Einstein’s theory of general relativity. This theory is based on a single axiom: ‘The principle of relativity’ (also known as the ‘principle of covariance’). It is the assertion that any law of nature

for any particular phenomenon, as expressed by any observer, with respect to a comparison in all possible reference frames relative to his or her own reference frame, must be in *one-to-one correspondence*. This is equivalent to saying that the laws of nature must be totally objective — independent of reference frames.

It might be argued that the theory of relativity, according to this definition, is a tautology rather than a science. For how could a law be a law, by definition, if it were not fully objective? It would be like saying: a human female is a woman. Of course this is a true statement, but it is only the definition of a word. (*It is called a 'necessary truth'.*) Nevertheless, the principle of relativity is not a tautology because it depends on two implicit assumptions that are not tautological. One is the idea that there are laws of nature in the first place. The second implicit assumption is that we can comprehend and express the laws of nature. These are not *necessary truths*. They are *contingent*, and thus qualify as scientific statements. (*This view supports Karl Popper's distinction between a necessary truth and a scientific truth.*¹⁰)

The assertion of the existence of laws of nature corresponds to saying that for every physical effect in the world, there is an underlying physical cause. Here, there is an implication that the universe is totally ordered. It is the scientist's obligation, then, to pursue this order in terms of the cause-effect relations — the laws of nature. If, as Galileo believed, it is impossible to achieve a *complete knowledge* of the order of the universe,¹¹ regarding the laws for any of its phenomena, it is still the obligation of *scientists* to pursue an *increase* in their knowledge and understanding, though never expecting to reach complete comprehension.

The Role of Space and Time

The second implicit assumption is that we can comprehend and express the laws of nature, in our own language. It is the latter assumption where the space and time measures come in. They are the 'words' of a language that we invent (not the only possible language!) for facilitating an expression of the laws of nature.



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It is important to recognize that the laws of nature, *per se*, are not identical with the language that we invent to express them. Nature is there, it is *of* the universe, whether or not we are there to express its characteristics. Our language, on the other hand, is invented by us, to help us express the laws of nature. Space and time are not physical things that can contract, distort, and so on. Thus, it is not correct to say that the existence of matter distorts spacetime, as a lead sphere may distort a rubber sheet that it falls into! Space and time are not more than the elements of a language we invent for the purpose of facilitating an expression of the laws of nature — such as the physical laws of the universe. This is a mathematical language that is continually perfected, perhaps for want of a better mathematical language to express the laws of nature.¹²

As we have discussed earlier, three space parameters and the time parameter in the laws of nature, according to relativity theory, do not have any objective (i.e. frame-independent) significance in themselves. What is significant as an objective language is the unification of space and time into spacetime. This means that purely space measures in the expression of a law of nature in one reference frame is a combination of space and time parameters in the expressions of the same law of nature in different reference frames. It then follows that the time measure must be expressed in the same units as the space measure. That is, in the different reference frames where the laws of nature are compared, instead of calling the time measure t seconds, t' seconds, t'' seconds. etc., they must be called ct centimeter, ct' centimeter, ct'' centimeter, etc. where c , as a conversion factor, must be frame-independent, with the dimension of centimeter/second — the dimension of a speed. Thus, the principle of relativity predicts there must be an invariant speed associated with the time measure. It turns out, in looking at one particular law of nature — the Maxwell field equations that underlie electromagnetism — that c is the invariant speed of light in a vacuum. (*In the initial stages of relativity theory, Einstein said that there were two independent axioms that underlie this theory: (1) the principle of relativity and (2) the invariance of the speed of*

light. We see here that the latter is not an independent axiom — it follows logically from the principle of relativity.)

The spacetime language forms a continuous set. The principle of relativity then requires that the transformations of the expressions of the laws of nature, such as the laws of cosmology, in one reference frame, to a *continuously connected* reference frame, must leave the form of the law unchanged. (It is called ‘covariance’.) This requirement implies that the laws themselves must be *continuous* field equations (as is the Maxwell formulation for electromagnetism) and that the solutions of these laws are *continuous fields* (everywhere). One further requirement of the transformations of relativity theory, according to the principle of relativity, is that discovered by Emma Noether, called ‘*Noether’s theorem*’.¹³ It is that a necessary and sufficient condition for the laws of conservation (of energy, momentum and angular momentum) to be included with the other laws of nature, in the special relativity limit of the theory, is that the transformations must not only be continuous, but also analytic. That is, their derivatives must exist to all orders. Singularities are then automatically excluded. The set of continuous, analytic transformations, everywhere, forms a *Lie group*. In the case of the theory of general relativity, this is the ‘Einstein group’. In the special relativity limit of this theory, this is the ‘Poincaré group’. These Lie groups underlie the algebraic logic of the theory of relativity. The functions that transform in this way, to maintain the principle of relativity, are called ‘regular’, indicating the inadmissibility of singularities anywhere in the universe! This would include a rejection of the ‘black hole’ (as it is commonly understood in physics today), and the singularity of the ‘big bang’ in cosmology. It is a condition on the solutions of general relativity, emphasized by Einstein throughout his lifelong pursuit of this theory. That is, Einstein would not have accepted either of these singularities, commonly discussed today, as realities!¹⁴ (*In personal discussions with Prof. Nathan Rosen, one of Einstein’s close collaborators in the 1930s, he acknowledged the validity of this statement.*)

The Lie group of the theory of general relativity, the Einstein group, has 16 essential parameters; these are the derivatives of

the four spacetime coordinates of one (primed) reference frame where the laws are expressed, with respect to those of a different (unprimed) spacetime reference frame, $\partial x^{\mu} / \partial x^{\nu}$, where μ and ν refer to the four space and time coordinates in each reference frame. The significance of the number 16 of essential parameters of the Lie group of general relativity theory is that there must be 16 independent field equations to prescribe the spacetime.¹⁵ What is interesting here is that Einstein already showed that there are (at least) 10 independent equations (the ‘Einstein field equations’), with 10 solutions, that underlie the gravitational phenomenon, and there are 6 independent solutions of the laws of electromagnetism — the three components of the electric field and the three components of the magnetic field. It has been shown that the Lie group of general relativity then implies a truly unified field theory, with a 16-component metrical field, where gravity and electromagnetism are unified in terms of a single field of force. This new 16-component metrical field of general relativity theory is a four-vector field $q^{\mu}(x)$, in which each of the four components is a quaternion rather than a real number field.¹⁶ *The new quaternion formulation of general relativity theory implies a cosmology in which the expansions and contractions of the oscillating model of the universe, as a whole, are spiral rather than isotropic, and where the Hubble law is predicted as a first approximation. This is discussed in Chapter 4 and in Ref. 17.*

Geometry and Matter

Einstein started his theory of general relativity with the idea that the variability of the matter content of the universe implies a variability of the coefficients of the metric tensor that underlies the geometry in the language of spacetime. This is tied to the idea that the spacetime is not more than a language that *reflects* the physical properties of the matter content of the universe. Analogous to the logic of ordinary verbal language, this language, in turn, has a geometrical logic and an algebraic logic. The algebraic logic is in terms of the underlying symmetry group of the theory, as discussed above. The geometrical logic is expressed in terms of the invariant differential metric of the

spacetime:

$$ds^2 = g^{\mu\nu}(x)dx_\mu dx_\nu = ds'^2,$$

where summation is implied over the subscripts and superscripts, and $\mu, \nu = 0, 1, 2, 3$ are the temporal (0) and spatial (1, 2, 3) coordinates. $g^{\mu\nu}(x)$ is the ‘metric tensor’. It is, in Einstein’s nonsingular field theory, a ‘regular function’, *everywhere*, and a second-rank, symmetric, 10-component tensor. Its continuous variability in the spacetime x is a *reflection* of the continuous variability of the matter fields of the spacetime. The idea is then that the spacetime transformations that keep the invariance of $ds^2 = ds'^2$ are those that transform *any* law of nature covariantly — i.e. preserving its form in all (continuously connected) reference frames. This is a *Riemannian geometrical system*.

The geodesic of the spacetime — the path of minimum separation between any two of its points — is determined by minimizing the invariant integral that is the path length $\int ds$. The physical significance of the geodesic is that it is the natural path of an unobstructed body. In Galileo’s classical view, the geodesic is a straight line; the family of such straight lines is a ‘flat spacetime’. *Galileo’s principle of inertia* then asserts the natural path of an unobstructed body is a straight line. In contrast, the geodesic of the Riemannian spacetime is a curve. Thus, the natural path of an unobstructed body in such a spacetime is a curve. The family of such curves is a ‘curved spacetime’. The variables of this curve, that gives it its structure, are determined by the variability of the matter of the spacetime. As the matter of the system is then continuously depleted, the curved spacetime approaches a flat spacetime. In this limit of a perfect vacuum, *everywhere*, the values of the metric tensor become:

$$g^{00} \rightarrow 1, \quad g^{kk} \rightarrow -1, \quad g^{\mu \neq \nu} \rightarrow 0, \quad (k = 1, 2, 3)$$

so that, for a perfect vacuum, *everywhere*, the invariant metric becomes:

$$ds^2 = (dx_0)^2 - (dx_1)^2 - (dx_2)^2 - (dx_3)^2 = ds'^2$$

This is indeed the invariant metric of special relativity theory — the ‘Lorentz metric’. Thus, the theory of special relativity, discovered originally to preserve the forms of the laws of nature in inertial frames of reference (i.e. frames in constant, rectilinear speed relative to each other) really only applies to the ideal case of the perfect vacuum. Where its formalism ‘works’ in physics, for a material medium (e.g. in its prediction of the energy-mass relation, $E = mc^2$, the dynamics of elementary particles, etc.) must then be a mathematical approximation for the metric of general relativity. This assumes the curvature of spacetime, in a small region, may be *approximated* by a flat spacetime, tangent to the curved spacetime where it is applied.

The idea of the theory of general relativity, as discussed earlier, is that the geometry of spacetime is to *reflect* the matter content of the closed system, in principle the universe. For example, the existence of the sun is *reflected* in the curved spacetime geometry in the vicinity of the sun, causing the trajectory of a beam of starlight to bend as it passes this region of space. This bending was predicted qualitatively and quantitatively by Einstein and it was then observed in agreement with his theory of general relativity. It is a gravitational effect that was not predicted by the earlier Newtonian theory of universal gravitation. Along with other gravitational effects not predicted by the classical theory, as well as giving back the equations of the classical theory *as an approximation*, Einstein’s theory of general relativity superseded Newton’s theory of universal gravitation, as a true *explanation* of gravity. Thus, Einstein discovered the field equations that predict the phenomenon of gravity. It relates the geometry of spacetime, in terms of the Riemannian metric tensor, $g^{\mu\nu}(x)$ and its changes in spacetime to the matter field variables of the closed system, chosen to be the energy-momentum tensor of its material content.

Generalization of Einstein’s Field Equations

Einstein’s field equations, in $g^{\mu\nu}(x)$, are 10 independent, nonlinear differential equations. But they are too few in number, for this reason: the symmetry group of general relativity theory — the ‘Einstein group’ — is the group of transformations that defines the

invariance of $ds^2 = g^{\mu\nu}dx_\mu dx_\nu = ds'^2$. This is a *Lie group*, a group of continuous, analytic transformations that preserves the forms of the laws of nature under the changes from one frame of reference to any continuously connected frame of reference. This preservation is the requirement of the underlying principle of relativity. The number of *essential parameters* of the Lie group, $\partial x^{\mu'}/\partial x^\nu$, is 16 in number. It implies that the most general form of the field equations, subject to the principle of relativity, must be 16 in number rather than 10.

Why are Einstein's equations 10 in number rather than 16? It is because the form of these equations is more symmetric than they need be, in accordance with the (16-parameter) Einstein group. They are covariant (form-invariant) with respect to the continuous transformations, as they must be. But they are also covariant with respect to the discrete reflections in space and time, which is not required. By lifting the space and time reflection transformations, Einstein's equations thereby *factorize* to 16 independent equations, as it is shown in Chapter 3 (pp. 27–36) and in Ref. 18.

It is important to note that the invariant differential metric of the spacetime is not ds^2 , it is ds . How does one take the square root of ds^2 ? It is usually stated that its square root is $\pm ds$, and the minus sign is simply thrown away! But this is not valid, as the square root is double-valued at all points of the spacetime, i.e. there is an ambiguity in the sign of this term, *everywhere*.

The answer to this question comes from the fact that the irreducible representations of the Einstein group of general relativity obey the algebra of quaternions (as well as the irreducible representations of the Poincaré group, of special relativity). The quaternion is a generalization of the complex number in 2-dimensional space, whose basis elements are 1 and i ($= \sqrt{-1}$), forming the complex function, $f(z = x + iy) = u(x, y) + iv(x, y)$.¹⁹ The basis elements $(1, i)$ of the complex number in two dimensions generalize to the four basis elements, σ^0, σ^k , (where $k = 1, 2, 3$); these are the unit two dimensional matrix and the three Pauli matrices. Thus, the quaternion has the *form* in the 4-dimensional space

$$q(x) = \sigma^0 x_0 + \sigma^1 x_1 + \sigma^2 x_2 + \sigma^3 x_3$$

We then define the quaternion four-vector $q^\mu(x)$, with the differential invariant quaternion of general relativity,

$$ds = q^\mu dx_\mu$$

Thus, there is no ambiguity here as to the sign of the invariant ds . It is important, and clear from this form of the quaternion, in terms of two-dimensional matrices that the product of two quaternions is not commutative under multiplication, i.e. $q_1 q_2 \neq q_2 q_1$.

Each of the four components of the four-vector q^μ is quaternion-valued and thus has four components. Thus, this quaternion metrical field has $4 \times 4 = 16$ components. This is a unique expression for the invariant $ds = q^\mu dx_\mu$. Further, the quaternion metric field transforms as a second rank spinor, that is, q^μ transforms as $(\psi \times \psi^*)^\mu$. Thus, the basis elements of the quaternion are two-component spinor variables ψ .

We see here that any law of nature, *whether in particle physics or in cosmology — the physics of the universe* — that is compatible with the symmetry required by relativity theory, in special or general relativity, must be in terms of spinor and quaternion variables. This is a requirement of the algebraic logic — the group structure — of the theory of relativity. It is the reason why Dirac's special relativistic theory of wave mechanics led to spinor degrees of freedom in the description of the electron (and a quaternion operator to determine these solutions). That is, the spin degrees of freedom in Dirac's electron equation are not a consequence of quantum mechanics, *per se*, as many have claimed! It is a consequence of the symmetry imposed by the theory of relativity.

The correspondence of the quaternion metric field and the metric tensor of Einstein's formulation is then in terms of the product of ds and its quaternion conjugate ds^* (corresponding to its time (or space) reflection):

$$ds ds^* \approx -(1/2)(q^\mu q^{\nu*} + q^\nu q^{\mu*}) dx_\mu dx_\nu = g^{\mu\nu} dx_\mu dx_\nu$$

Thus, the quaternion formulation in general relativity, $ds = q^\mu dx_\mu$, is a factorization of the metric tensor formulation of the standard Einstein theory.

Similar to Einstein's derivation of the field equations in $g^{\mu\nu}$ from a variational principle, the factorized field equations in q^μ may be derived from a variational principle. These derivations are shown in Ref. 16. The quaternion form yields 16 field relations that replace the 10 relations of Einstein's symmetric tensor form of his field equations in general relativity. In addition to explaining gravity, as will be described in Chapter 3 (pp. 27–36), the quaternion form predicts new physical effects in the cosmological problem of the universe as a whole. One important new feature is the torsion of spacetime. It predicts, as examples, the rotation of the galaxies and the Faraday Effect regarding the propagation of cosmic electromagnetic radiation, i.e. the rotation of the plane of polarization of this radiation as it propagates throughout the universe. Both are observed astrophysical effects, not predicted by the standard Einstein tensor formulation. A further prediction is an anisotropic expansion and contraction of the universe, in a spiral fashion. Another important difference is that the geodesic equation, that prescribes a natural motion along a curve of an unobstructed body, has a quaternion form. That is, to prescribe the motion of a body along a trajectory, parameterized by the time measure, one must have four parameters, rather than one, to prescribe the time change, as a body moves from one spatial location along its trajectory to another. This is the generalization of the time parameter in physics theorized by William Hamilton, from his discovery of the quaternion algebra in the 19th century.

A Unified Field Theory

By iterating the 16 field equations in q^μ with the conjugated solution $q^{\nu*}$ on the left, and iterating the conjugated (i.e. reflected) equation in $q^{\nu*}$ on the right with q^μ , second-rank tensor equations are generated. Adding these two iterated equations generates a symmetric second-rank tensor equation (10 components) that we will see is in one-to-one correspondence with Einstein's original tensor equations (Chapter 3), thus explaining gravity. Subtracting these equations generates an antisymmetric second-rank tensor equation

that can then be put into one-to-one correspondence with Maxwell's equations, thus explaining electromagnetism. Thus, the original 16 quaternion metrical field equations break up into 10 equations that explain gravity and 6 equations that explain electromagnetism — *this is the unified field theory that was sought by Einstein*. In this field theory, in general relativity both physical phenomena are incorporated in the single, 16-component quaternion field q^μ . It is this formal, generalized expression of general relativity that generates a new cosmology,¹⁷ relating to the physics of the universe. The dynamics is an oscillating universe, between expansions and contractions, in spiral configuration. As will be discussed in Chapter 4 (pp. 37–50), the Hubble law is an approximation for this dynamics, over sufficiently short times in the expansion phases of the universe.

In the next chapter, we will discuss the physics and outline the formal development of Einstein's tensor form of general relativity theory as a language of cosmology.

A Language of Cosmology: The Mathematical Basis of General Relativity

Introduction

The basis of the theory of general relativity, as underlying the physics of the cosmos, was discussed in Chapter 1. In this chapter we will outline the mathematical derivation of this formalism, as it was shown in Einstein's study. A generalization will then be shown in the next chapter from Einstein's tensor formulation to the quaternion formulation. This leads from the 10 components of the symmetric tensor solutions to 16 independent components of the quaternion solution for the metric of spacetime.¹⁶ In Chapter 3, we will utilize the quaternion formulation to demonstrate (1) a unification of electromagnetism with gravity, (2) an oscillating universe cosmology, and (3) as a first approximation, the Hubble law, indicating the expansion of the universe. In Chapter 4, the spiral configuration of the expanding and contracting universe will be demonstrated explicitly. As a bonus, it will be shown in Chapter 5 that the spiraling matter of the oscillating universe provides a mechanism for the creation of magnetic fields of opposite polarity that separate out matter from antimatter of the universe at the beginning of each expansion phase of the universe.

Einstein's Tensor Formulation²⁰

The invariant differential metric with the Riemannian geometry of spacetime is:

$$ds^2 = g^{\mu\nu}(x)dx_\mu dx_\nu = ds'^2 = g^{\mu\nu'}(x')dx'_\mu dx'_\nu \quad (2.1)$$

where the summation convention is assumed over the space and time coordinates, $\mu, \nu = 1, 2, 3, 0$.

The metric tensor $g^{\mu\nu}(x)$ is a regular (continuous and analytic) field, *everywhere*, as has been argued in the preceding chapter. The geodesics of this spacetime, i.e. the paths wherein the distance between their points is a minimum, are *variable curves*, rather than the straight lines of a Euclidean spacetime.

The physical significance of the geodesic is that it is the natural path of an unobstructed body. To remove the body from this path would require the input of external energy. The family of such curved geodesics is called a 'curved spacetime'. In the limit, as the matter of a closed system (in principle, the universe) is depleted toward a vacuum, *everywhere*, the curved spacetime approaches a 'flat spacetime'. In this limit, the invariant Riemannian metric (2.1) would become the invariant Euclidean (Lorentz) metric of special relativity,

$$ds^2 = (dx^0)^2 - dr^2 = ds'^2 = (dx^0')^2 - dr'^2 \quad (2.1')$$

The principle of relativity then requires that the same spacetime transformations that leave ds^2 invariant (Eq. (2.1) or (2.1')) must leave all of the laws of nature *covariant* (i.e. frame- independent).

The Riemann Curvature Tensor

As we have discussed earlier, it is the thesis of the theory of general relativity that the geometry of spacetime is to *reflect* the matter content of the universe. Einstein discovered the field equations that express this reflection of matter in geometry. The left-hand side of his equations is: $G_{\mu\nu} = R_{\mu\nu} - (1/2)g_{\mu\nu}R$. This is called 'the Einstein Tensor'. As it will be shown below, the Ricci tensor, $R_{\mu\nu}$, also entails

the geometrical field, the metric tensor $g_{\mu\nu}$. The right-hand side of Einstein's equations is chosen to be the energy-momentum tensor $T_{\mu\nu}$, representing the material content of the universe. The tensor $R_{\mu\nu}$, and the 'scalar curvature', R , will be defined explicitly below. The origin of the left-hand side of Einstein's equations, the Einstein tensor, $G_{\mu\nu}$, is in the Riemannian curvature tensor $R^\lambda_{\mu\nu\rho}$, that is defined in terms of a difference of second covariant derivatives of a vector field, V_μ , as follows:

$$V_{\mu;\nu;\rho} - V_{\mu;\rho;\nu} = R^\lambda_{\mu\nu\rho} V_\lambda \quad (2.2)$$

As the flat spacetime is asymptotically approached (i.e. toward a perfect vacuum *everywhere*), $R^\lambda_{\mu\nu\rho} \rightarrow 0$ and the covariant, second derivatives of the vector field V_μ become the ordinary second derivatives of this field, which do not depend on the order of differentiation.

The Riemann curvature tensor $R^\lambda_{\mu\nu\rho}$ is derived from the covariant derivatives of the four-vector field V_ν as follows:

$$V_{\nu;\lambda} = \partial_\lambda V_\nu - \Gamma^\rho_{\lambda\nu} V_\rho \quad (2.3)$$

where the coefficients of the 'affine connection' are¹⁶:

$$\Gamma^\rho_{\mu\alpha} = (1/2)g^{\rho\lambda}(\partial_\mu g_{\lambda\alpha} + \partial_\alpha g_{\mu\lambda} - \partial_\lambda g_{\alpha\mu}) \quad (2.4)$$

The combination of eqs. (2.2) and (2.3) then gives the Riemannian curvature tensor in terms of the affine connection as follows:

$$R^\lambda_{\mu\nu\rho} = \partial_\nu \Gamma^\lambda_{\mu\rho} - \partial_\rho \Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\alpha\nu} \Gamma^\alpha_{\mu\rho} - \Gamma^\lambda_{\alpha\rho} \Gamma^\alpha_{\mu\nu} \quad (2.5)$$

The Ricci tensor, in turn, is a contraction of the Riemann curvature tensor, as follows:

$$R_{\mu\nu} = R^\lambda_{\mu\nu\lambda} = \partial_\nu \Gamma^\lambda_{\mu\lambda} - \partial_\lambda \Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\alpha\nu} \Gamma^\alpha_{\mu\lambda} - \Gamma^\lambda_{\alpha\lambda} \Gamma^\alpha_{\mu\nu} \quad (2.6)$$

It is readily shown¹⁶ that the Ricci tensor is symmetric, $R_{\mu\nu} = R_{\nu\mu}$.

The scalar curvature is, by definition, the contraction

$$R = g^{\mu\nu} R_{\mu\nu} \quad (2.7)$$

Because the covariant divergence of the energy momentum tensor vanishes, $T^{\mu\nu}_{;\nu} = 0$ (the flat space limit of this law is $\partial^\nu T_{\mu\nu} = 0$ — the

conservation law of energy and momentum), the covariant divergence of the left-hand side of Einstein's equations, that is, of the Einstein tensor $G_{\mu\nu}$, must vanish. Analysis indicates¹⁶ that $R_{\mu\nu}^{;\nu} = (1/2)(g_{\mu\nu}R)^{;\nu}$. Thus, $G_{\mu\nu}^{;\nu} = (R_{\mu\nu} - (1/2)g_{\mu\nu}R)^{;\nu} = 0$, so that we have the Einstein field equation:

$$G_{\mu\nu} \equiv R_{\mu\nu} - (1/2)g_{\mu\nu}R = kT_{\mu\nu} \quad (2.8)$$

wherein the covariant divergence of both sides of this equation vanish.

The derivation of Einstein's field equations (2.8) from a variational principle is demonstrated in Reference 16.

Summing up, the field equations (2.8) with (2.4), (2.5) and (2.6) are 10 nonlinear differential equations, whose solutions are the components of the metric tensor $g_{\mu\nu}$. Once these solutions are determined, they may be inserted into the geodesic equation, shown below, to determine the path of an unobstructed body in the curved spacetime.

The Geodesic Equation

The geodesic equation follows from a minimization of the indefinite line integral $\delta \int ds = 0$. This yields the geodesic 'equation of motion'²¹:

$$d^2x^e/ds^2 + \Gamma_{\mu\nu}^e(dx^\mu/ds)(dx^\nu/ds) = 0 \quad (2.9)$$

Thus, with the affine connection coefficients in terms of the metric tensor (2.4), the Ricci tensor (2.6) and the scalar curvature R (2.7), the solutions $g_{\mu\nu}$ of Eq. (2.8) in Eq. (2.9) yield the geodesic path of a body in the curved spacetime. With the foregoing, we then have the apparatus to determine the explicit effects of gravitational phenomena.

The Vacuum Equation

The actual numerical successes of the formalism of general relativity, thus far, come from the assumption of a vacuum, $T_{\mu\nu} = 0$, *everywhere*, yielding the nonlinear differential equation that is the 'vacuum

equation':

$$R_{\mu\nu} = 0 \quad (2.10)$$

The assumption made here is that $T_{\mu\nu} \neq 0$ inside of the surface of a star, yielding the non-homogeneous equation (2.8) there, but that it is zero outside of the star, yielding the 'outside' homogeneous equation (2.10). One then *adds* the two solutions to yield the general solution, and match the solution of the homogeneous equation to the solution of the non-homogeneous equation at the surface of the star to determine the integration constants, as it is done in classical (linear) physics. However, this is not valid here, in principle, since all fields in general relativity solve nonlinear equations, thus they are not additive *anywhere*, resulting in a different solution. There cannot be a sharp cut-off, where the *continuous field* $T_{\mu\nu}$ changes from a nonzero value, as inside of a material medium (a star), to a zero value, outside of the star. This is because of the nonlinear mathematical structure of the field theory. There is, in fact, no objective 'inside' and 'outside' of a body, such as a star. That is, the general solution of a nonlinear differential equation is *not* the sum of a homogeneous vacuum solution, where there is zero on the right (the outside of the star) and a solution for non-zero on the right (the inside of the star). It then follows that in general relativity we must interpret the vacuum equation (2.10) as an *approximation* for the actual field equation of general relativity in Einstein's tensor formulation (2.8).

The Schwarzschild Solution²²

With the definition of the Ricci tensor in (2.6) and the affine connection in (2.4), the nonlinear differential equation for the vacuum (2.10) in $g_{\mu\nu}$ was shown by Schwarzschild to yield the solution:

$$ds^2 = (1 - 2\alpha/r)(dx^0)^2 - dr^2/(1 - 2\alpha/r) - r^2 d\Omega^2 \quad (2.11)$$

where α is a constant of the integration (with the dimension of length) and $d\Omega$ is the differential solid angle. Comparison with the limit, in which the Newtonian theory of gravity appears as an

approximation, yields: $\alpha = GM/c^2$, where G is the gravitational constant in Newton's force law: $F = Gm_1m_2/r^2$, (m_1 and m_2 are the gravitationally interacting masses and r is their mutual separation), and M is the mass of the body that gives rise to the metric (2.11). We see there is a singularity in this solution at the radial distance $r = 2\alpha$. (For the sun, this is the order of a few kilometers from its center — well inside the body of the sun).

In view of Einstein's admission of only nonsingular (regular) solutions of general relativity, this singularity is not real. When the vacuum equation (2.10) is extended to the full form of Einstein's field equations (2.8), according to their meaning in Einstein's view, the apparent singularity washes away.

The black hole

Nevertheless, for those who see (2.11) as an exact solution of general relativity, there is a peculiar type of star predicted, if the matter of the sun could be condensed to a sphere with the Schwarzschild radius, $r = 2\alpha$ (a diameter of a few kilometers for the sun) without blowing apart before reaching this dimension and enormous density! The singularity at this 'event horizon' would prevent any signal from propagating away from the star. This type of star is called a 'black hole'.²³

As mentioned above, this singular solution is inadmissible in view of our acceptance of only regular (nonsingular) solutions of the equations of general relativity. Still, there is a possibility of a 'black hole' in a different context. If a star is sufficiently dense that the geodesics associated with it are *closed*, then all emissions from this star, including light and gravitation, that would be propagated away from the star would then return to it, along the closed geodesic, and thus reabsorbed by the same star. It would then be black to any outside observer. Such a star may then also be called a 'black hole'. *The possibility of the existence of such a star would depend on the existence of 'stable solutions' associated with the closed geodesics. This has not yet been established, theoretically, nor is there yet any conclusive experimental proof for the existence of a black hole.*

The Crucial Tests of General Relativity

The Schwarzschild solution (2.11) of the vacuum equation (2.10) leads to the three crucial tests of the theory of general relativity. The first is the perihelion precession of Mercury's orbit.

This effect was observed in the 19th century, but unexplained with classical physics. It was predicted, both qualitatively and quantitatively, correctly in the 20th century by general relativity.

The second effect was the bending of a beam of starlight as it propagates past the rim of the sun. This was observed in 1919 by a group of astronomers, headed by A. Eddington, from Cambridge University. Their observations were in agreement with the predictions of general relativity, qualitatively and quantitatively.

The third crucial test was the gravitational red shift. This is a shift of frequencies of emitted radiation in the visible spectrum toward the red end of the spectrum, as the potential energy environment of the emitter is increased. This was observed empirically in the 1950s by R.V. Pound and his co-workers at Harvard University. It was in qualitative and quantitative agreement with the prediction of general relativity.²⁴

In addition, (2.10) also predicts, in a linear approximation, Newton's gravitational equation. The foregoing then constitutes the success of Einstein's theory of general relativity to supersede Newton's theory, to explain the phenomenon of gravity.²⁵

It was mentioned in the preceding chapter that, in the 17th century, Newton himself was not satisfied that his theory was indeed an *explanation* of the phenomenon of gravity, though he understood it as an adequate *description*. His problem was with the role of 'action-at-a-distance' in the theory.

Still, Newton was willing to use the concept because it 'worked'. He said that he did not form hypotheses. For this reason, Newton did not believe that his theory *explained* the phenomenon of gravity, though he did believe that it correctly *described* it, in his day. It was not, indeed, *explained* until the phenomenon of gravity was correctly predicted by Einstein's theory of general relativity in the early part of the 20th century.

On the use of the vacuum equation (2.10) to accurately describe gravity, it is not to be interpreted, according to Einstein's meaning, as an exact form of the theory of general relativity. For, as we have discussed earlier, the geometry of spacetime, expressed in terms of the metric tensor, is to *reflect* the continuous matter content of the physical system; in principle, the universe. It then follows, logically, that if there would be no matter content of the universe, that is, if the universe were a perfect vacuum, *everywhere*, there would be no variable geometry to reflect this vacuum except for the flat spacetime, with the Lorentzian metric, $g^{00} = 1$, $g^{11} = g^{22} = g^{33} = -1$, and $g^{\mu \neq \nu} = 0$. In this case, $ds^2 = (dx^0)^2 - dr^2$. This is the invariant metric of the theory of special relativity theory. We see, at this juncture, that the theory of special relativity is only an exact theory for the ideal case where the entire universe would be a perfect vacuum! Yet, it is a valid *approximation* for the formalism of general relativity under special physical conditions (as in its use in particle physics), when matter is sufficiently rarefied and it does not move relative to other matter at speeds close to the speed of light. *Present-day experiments at the laboratory, CERN, in Switzerland, using the 'supercollider', are attempting to duplicate the quantities of matter density and energy transfer near the time of the (last) 'big bang'. The results of this experimentation may indeed lead to a refutation of the use of special relativity and the need for the mathematical formulation of general relativity to explain these data.*

The Logic of the Spacetime Language

The logic of the spacetime language of general relativity is in two parts: geometry and algebra. The geometrical logic relates points to points and lines in the sense of congruence, parallelism, etc. The algebraic logic relates points to points and lines in the sense of rules of combination, associativity, commutativity, etc. The algebraic logic is expressed most compactly in terms of a symmetry group. *(Modern Mathematics has shown that all of the theorems of geometry and algebra may be merged into a common set of theorems. However, for our purposes, we will discuss them separately.)*

The symmetry group of general relativity theory — the ‘Einstein group’ — is a set of *continuous transformations* that leave the Riemannian metric (2.1) invariant. Requiring further, regular solutions, these transformations must not only be continuous, but also analytic. The origin of this requirement is Noether’s theorem.¹³ It requires that a necessary and sufficient condition for the incorporation of the conservation laws — of energy, momentum and angular momentum in the special relativity limit of the field theory — is the analyticity of the field variables and their transformations. That is, the derivatives of the space and time coordinates of one reference frame with respect to the space and time coordinates of any continuously connected reference frame must exist to all orders. All such transformations are then nonsingular. Thus, the Einstein group is, by definition, a ‘Lie group’. The number of *essential parameters* of this group is 16. These are the derivatives of one set of spacetime coordinates, of one reference frame, with respect to another set of spacetime coordinates, of a different (continuously connected) reference frame, $\partial x^{\mu}/\partial x^{\nu}$.

Why, then, are Einstein’s equations (2.8), 10 in number rather than 16? It is because they are more symmetric than they need be. These field equations are covariant with respect to the continuous changes in space and time, as they are required to be. But they are also symmetric with respect to the discrete reflections in space and time, which is not a requirement. Indeed, the Einstein group is a *continuous group*, without any discrete reflection transformations.

We will see in the next chapter that by dropping the reflection symmetry elements from the Einstein equations (2.8), they factorize to a new form wherein the symmetric tensor field variables $g^{\mu\nu}$ (10 components) are replaced with the quaternion field q^{μ} . This is a four-vector field, but each of the four components is quaternion-valued, with four independent components, rather than real-number-valued. Thus, this metric field has $4 \times 4 = 16$ components. This quaternion field equation for general relativity is then covariant under the continuous transformations in space and time but it is not covariant with respect to the discrete reflections in space and time.