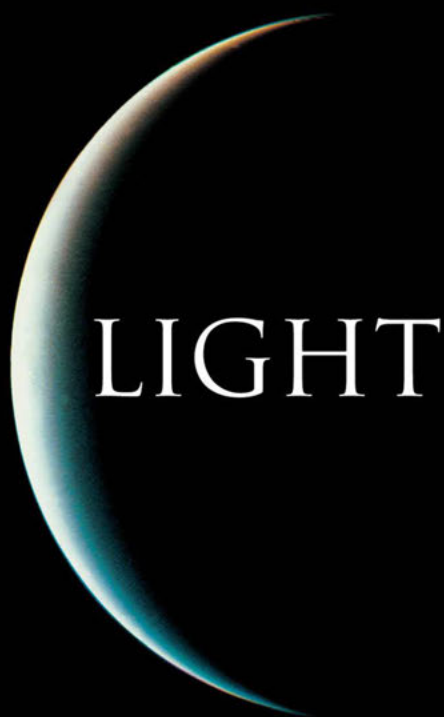


A STORY OF



LIGHT

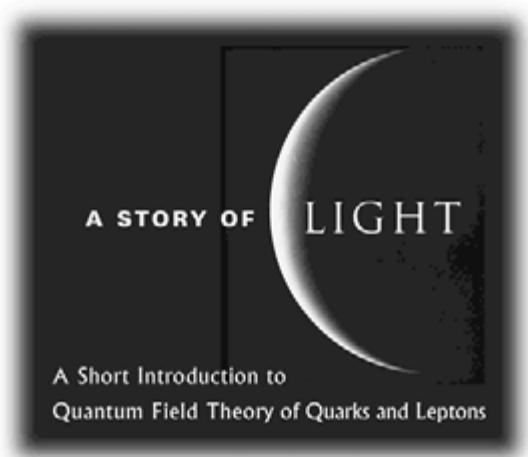
A Short Introduction to
Quantum Field Theory
of Quarks and Leptons

M. Y. HAN



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A Short Introduction to
Quantum Field Theory
of Quarks and Leptons

M. Y. HAN

Duke University, USA

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A STORY OF LIGHT

An Introduction to Quantum Field Theory of Quarks and Leptons

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For Eema, Grace and Leilani

Acknowledgments

I would like to thank my students who insisted that I write this book after my lectures on the developments in quantum field theory. I would also like to thank Dr. Jaebeom Yoo, a postdoctoral research associate, and Mr. Chang-Won Lee, a graduate student in the Physics Department of Duke University, for valuable discussions and technical help in the preparation of the manuscript. As with my previous book, *Quarks and Gluons*, the constant encouragement from Dr. K.K. Phua, Chairman of World Scientific Publishing Co. is gratefully acknowledged. Thanks are also due to the dedicated help of Ms. Lakshmi Narayan, a Senior Editor of World Scientific, who provided steady and patient guidance toward the completion of this book.

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Prologue

The relativistic quantum field theory, or quantum field theory (QFT) for short, is the theoretical edifice of the standard model of elementary particle physics. One might go so far as to say that the standard model *is* the quantum field theory. Having said that as the opening statement of this book, we must be mindful that both quantum field theory and the standard model of elementary particle physics are topics that are not necessarily familiar to many individuals. They are subject areas that are certainly not familiar to those outside the specialty of elementary particle physics, and in some cases not too well grasped even by those in the specialty.

The Standard Model of elementary particle physics is a term that has come into prominence as it became the paradigm of particle physics for the last three decades. In brief, the standard model aims to understand and explain three of the four fundamental forces — the electromagnetic, strong nuclear and weak nuclear — that define the dynamics of the basic constituents of all known matter in the universe.¹ As such, it consists of two interrelated parts: the part

¹The fourth force of nature, gravity, does not come into play in the scale of the mass of elementary particles and is not included in the standard model. Adapted from [1].

that deals with the question of what are the basic building blocks of matter and the second part concerned with the question of what is the theoretical framework for describing the interactions among these fundamental constituents of matter.

A century after the original discovery of quantum of light by Max Planck in 1900 and its subsequent metamorphosis into photon, the zero-mass particle of light, by Albert Einstein in 1905, we have come to identify the basic constituents of matter to be quarks and leptons — the up, down, strange, charm, top and bottom, for quarks, and the electron, muon, tauon, electron-type neutrino, muon-type neutrino, and tauon-type neutrino, for leptons. The three forces are understood as the exchange of “quanta” of each force — photons for the electromagnetic force, weak bosons for the weak nuclear force, and gluons for the strong nuclear force. These particles, some old, such as photons and electrons and some relatively new, such as the top and bottom quarks or the tauons and their associated neutrinos, represent our latest understanding of what are the basic constituents of known matter in the universe.

There are scores of books available which discuss the basic particles of matter, at every level of expertise. For a general readership, we can mention two books that contain no or very little mathematics, *Quarks and Gluons* by myself and *Facts and Mysteries in Elementary Particle Physics* by Martinus Veltman.²

The theoretical framework for the three forces or interactions is quantum field theory, that is, the relativistic quantum field theory. Each force has its own form, and again, some old and some new. Quantum electrodynamics, QED for short, was fully developed by the end of the 1940s and is the oldest — and more significantly, the only truly successful quantum field theory to date — of the family. Quantum chromodynamics, QCD, is the framework for the strong nuclear force that is mediated by exchanges of gluons. It was initiated

to merge gravity with the standard model have spawned such ideas as the grand unified theory, supersymmetry, and superstring, the so-called theory of everything. These topics are not discussed in this book.

² *Quarks and Gluons* by M. Y. Han, World Scientific (1999); *Facts and Mysteries in Elementary Particle Physics* by Martinus Veltman, World Scientific (2005).

in the 1960s and has been continually developed since, but it is far from becoming a completely successful quantum field theory yet. The theory for the weak nuclear force, in its modern form, was also started in the 1960s, and in the 1970s and 1980s, it was merged with quantum electrodynamics to form a unified quantum field theory in which the two forces — the electromagnetic and weak nuclear — were “unified” into a single force referred to as the electroweak force. Often this new unified theory is referred to as the quantum flavor dynamics, QFD. Thus, the quantum field theory of the standard model consists of two independent components — quantum chromodynamics and quantum flavor dynamics, the latter subsuming quantum electrodynamics.

Despite the abundant availability of books, at all levels, on basic building blocks of matter, when it comes to the subject of relativistic quantum field theory, while there are several excellent textbooks at the graduate level, few resources are available at an undergraduate level. The reason for this paucity is not difficult to understand. The subject of quantum field theory is a rather difficult one even for graduate students in physics. Unless a graduate student is interested in specializing into elementary particle physics, in fact, most graduate students are not required to take a course in quantum field theory. It is definitely a highly specialized course. Quantum field theory thus remains, while a familiar term, a distant topic. Many have not had the opportunity to grasp what the subject is all about, and for those with some rudimentary knowledge of physics at an undergraduate level beyond the general physics, the subject lies well beyond their reach.

The main purpose of this book is to try to fill this gap by bringing out the conceptual understanding of the relativistic quantum field theory, with minimum of mathematical complexities. This book is not at all intended to be a graduate level textbook, but represents my attempt to discuss the essential aspects of quantum field theory requiring only some rudimentary knowledge of the Lagrangian and Hamiltonian formulation of Newtonian mechanics, special theory of relativity and quantum mechanics.

There is another theme in this book and it is this. Throughout the course of development of quantum field theory, from the original quantum electrodynamics in which the Planck–Einstein photon

is deemed as the natural consequence of field quantization to the present-day development of the gauge field theory for quarks and leptons, the theories of electromagnetic field have been — and continue to be — a consistently useful model for other forces to emulate. In this process of emulating theories of electromagnetic field, the concept of particles and fields would go through three distinct phases of evolution: separate and distinct concepts in classical physics, the particle-wave duality in quantum mechanics, and finally, particles as the quanta of quantized field in quantum field theory. As we elaborate on this three-stage evolution, we will see that the photon has been — and continues to be — the guiding light for the entire field of relativistic quantum field theory, the theoretical edifice of the standard model of elementary particle physics.

.....

1

Particles and Fields I: Dichotomy

One may have wondered when first learning Newtonian mechanics, also called the classical mechanics, why the concept of a field, the force field of gravity in this case, is hardly mentioned. One usually starts out with the description of motion under constant acceleration — the downward pull of gravity with the value of 9.81 m/s^2 . Even when the universal law of gravity is discussed, for example, to explain the Kepler's laws, we do not really get into any detailed analyses of the force field of gravity.

In classical mechanics the primary definition of matter is the point mass, and the emphasis is on the laws of motion for point masses under the influence of force. The focus is on the laws of motion rather than the nature of force field, which is not really surprising when we consider the simplicity of the terrestrial gravitational force field — uniform and in one parallel direction, straight down toward the ground. A point mass is an abstraction of matter that carries mass and occupies one position at one moment of time and this notion of a point mass is diagonally opposite from the notion of a field, which, by definition, is an extended concept, spread out over a region of space.

As we proceed from the study of classical mechanics to that of classical electromagnetism, we immediately notice a big change: from

day one it is all about fields. First the electric field, then the magnetic field, and then the single combined entity, the electromagnetic field. No sooner than the Coulomb's law is written down, one defines the electric field and its spatial dependence is determined by Gauss' Law. Likewise, Ampere's Law determines the magnetic field and finally the laws of Faraday and Maxwell lead to the spatial as well as temporal dependence of electromagnetic field.

This dichotomy of the concept of point particle and that of field is in fact as old as the history of physics. From the very beginning, back in the 17th century, there were two distinct views of the physical nature of light. Newton advocated the particle picture — the corpuscular theory of light — whereas Christian Huygens advanced the wave theory of light. For some time — for almost a century and half — these two opposing views remained compatible with what was then known about light — refraction, reflection, lenses, etc. Only when in 1801 Thomas Young demonstrated the wave nature of light by the classic double-slit interference experiment, with alternating constructive and destructive interference patterns, the wave theory triumphed over the particle theory of light.

One might have wondered why the notion of field did not play a prominent role in the initial formulation of Newtonian mechanics, especially since both the gravitational force law and the Coulomb's law obey the identical inverse square force law:

$$F = G \frac{m_1 m_2}{r^2} \quad \text{for gravity}$$

and

$$F = k \frac{q_1 q_2}{r^2} \quad \text{for Coulomb's law}$$

where G and k are the respective force constants, m is mass and q is the electric charge.

The disparity is simply a practical matter of scale. At the terrestrial level, in our everyday world, the inverse square law really does not come into play; the curvature of the surface of the earth is approximated by a flat ground and the gravitational force lines directed toward the center of the earth become, in this approximation, parallel lines pointing downward. In this scale of things, the

field aspect of gravity is just too simple to be taken into account. There is no need to bring in any analyses of the gravitational field in the flat surface approximation.

On the contrary, with electric and magnetic forces, we notice and measure in the scale of tabletop experiments the spatial and temporal variations of these fields. The gradients, divergences and curls, to use the language of differential vector calculus, of the electric and magnetic fields come into play in the scale of the human-sized world and this is why the study of electromagnetism always starts off with the definition of electric and magnetic fields.

This well-defined dichotomy of particles and fields, diagonally opposite concepts in classical physics, would evolve through many twists and turns in the twentieth century physics of relativity and quantum mechanics, ending up eventually with the primacy of the concept of field over that of particle in the framework of quantum field theory.

The process of evolution of the concepts of particles and fields have taken a quite disparate path. The Newtonian mechanics has evolved through several steps, some quite drastic. First, there was the Lagrangian and Hamiltonian formulation of mechanics. One of the most important outcomes of this formalism is the definition of what is called the canonically conjugate momentum and this would pave the way for the transition from classical mechanics to quantum mechanics. Quantum field theory could not have developed had it not been the idea of canonically conjugate momentum defined within the Lagrangian and Hamiltonian formalism. As quantum mechanics is merged with special theory of relativity, the culmination of the particle view was reached in the form of relativistic quantum mechanical wave equations, such as the Klein–Gordon and Dirac equations, wherein the wavefunction solutions of these equations provide the relativistic quantum mechanical description of a particle. (More on these equations in later chapters.)

In contradistinction to this development of particle theory, the field view of classical electromagnetism remained almost totally unmodified. The equation of motion for charged particles in an electromagnetic field is naturally accommodated in the Lagrangian and

Hamiltonian formalism. In the Lagrangian formulation of classical mechanics, Maxwell's equations find a natural place by being one of the few examples of what is called the velocity-dependent potentials (more on this in the next chapter). The very definition of the canonically conjugate momentum for charged particles to be the sum of mechanical momentum and the vector potential of the electromagnetic field, discovered back in the 19th century, is in fact the foundation for quantum electrodynamics of the 20th century.

The contrast between the mechanics of particles and the field theory of electromagnetic fields becomes sharper when dealing with the special theory of relativity. The errors of Newtonian mechanics at speeds approaching the speed of light are quite dramatic, and of course, the very foundation of mechanics had to be drastically modified by the relativity of Einstein. Maxwell's equations for the electromagnetic field, on the other hand, required no modifications whatsoever at high speeds; the equations are valid for all ranges of speeds involved, from zero to all the way up to the speed of light. At first, this may strike as quite surprising, but the fact of the matter is that Maxwell's equations lead directly to the wave equations for propagating electromagnetic radiation — light itself. Maxwell's theory of the electromagnetic field is already fully relativistic and hence need no modifications at all.

The development of relativistic quantum mechanics demonstrates quite dramatically the primacy of the classical field concept over that of particles. To cite an important example, in relativistic quantum mechanics, the first and foremost wave equation obeyed by particles of any spin, both fermions of half-integer spin and bosons of integer spin, is the Klein–Gordon equation. Fermions must also satisfy the Dirac equation in addition to the Klein–Gordon equation (more on this in later chapters).

For a vector field $\phi_\mu(x)$ [$\mu = 0, 1, 2, 3$] for spin one particles with mass m , the Klein–Gordon equation is¹

$$(\partial_\lambda \partial^\lambda + m^2)\phi_\mu(x) = 0$$

¹Notations and the natural unit system are given in Appendix A and B.

where

$$\partial_\lambda \partial^\lambda = \frac{\partial^2}{\partial t^2} - \nabla^2.$$

For the special case of mass zero particles, of spin one, the Klein–Gordon equation reduces to

$$\partial_\lambda \partial^\lambda \phi_\mu(x) = 0.$$

The classical wave equation for the electromagnetic four-vector potential $A_\mu(x)$, on the other hand, in the source-free region is

$$\partial_\lambda \partial^\lambda A_\mu(x) = 0.$$

An equation for a zero-mass particle of spin one (photon) in relativistic quantum mechanics turns out to be none other than the classical wave equation for the electromagnetic field of the 19th century that predates both relativity and quantum physics!

2

Lagrangian and Hamiltonian Dynamics

Lagrange's equations were formulated by the 18th century mathematician Joseph Louis Lagrange (1736–1813) in his book *Mathématique Analytique* published in 1788. In its original form Lagrange's equations made it possible to set up Newton's equations of motion, $F = dp/dt$, easily in terms of any set of generalized coordinates, that is, any set of variables capable of specifying the positions of all particles in the system. The generalized coordinates subsume the rectangular Cartesian coordinates, of course, but also include angular coordinates such as those in the plane polar or spherical polar coordinates. The generalized coordinates also allow us to deal easily with constraints of motion, such as a ball constrained to move always in contact with the interior surface of a hemisphere; the forces of constraints do not enter into the description of dynamics. As originally proposed, the Lagrange's equations provided a convenient way of implementing Newton's equations of motion.

Lagrange's equations became much more than just a powerful addition to the mathematical technique of mechanics when about 50 years later, in 1834, they became an integral part of Hamilton's principle of least action. Hamilton's principle represents the mechanical form of the calculus of variations that covers wide-ranging fields



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of physics. Lagrangian and Hamiltonian formulation of mechanics that established the basic pair of dynamical variables — position and momentum — is the precursor to the development of quantum mechanics and when it comes to the development of quantum field theory Lagrangian equations play an absolutely essential role.¹

For a thorough discourse on the principle of least action in general, and the Hamilton's principle in particular, we will refer readers to many other excellent books on the subject. For our purpose we will focus on specific portions of the Lagrangian and Hamiltonian dynamics that describe the charged particles under the influence of an electromagnetic field. Not always fully appreciated, the Lagrangian and Hamiltonian descriptions of the electromagnetic interaction of the charged particles provide the foundation for quantum electrodynamics, and by extension, for the formulation of the quantum field theories of nuclear forces. The very origin of the field theoretical treatment of electromagnetic interaction traces its root to the classical Lagrangian and Hamiltonian dynamics.

The simplest way to show the equivalence of Lagrange's and Newton's equations is to use the rectangular coordinates, say, x_i ($i = 1, 2, 3$ for more conventional x, y, z). Using the notation $\dot{p} = dp/dt$ and $\dot{x} = dx/dt$, Newton's equations are

$$F_i = \dot{p}_i$$

$$p_i = m\dot{x}_i = \frac{\partial}{\partial \dot{x}_i} \left(\frac{1}{2} m \dot{x}_j^2 \right) = \frac{\partial T}{\partial \dot{x}_i}$$

where T is the kinetic energy.

For a conservative system

$$F_i = -\frac{\partial V}{\partial x_i}$$

¹Many excellent standard textbooks on classical mechanics include rich discussions on these subjects — Hamilton's principle, Lagrange's equations, and the calculus of variations. At the graduate level, the *de facto* standard on the subject is *Classical Mechanics* by Herbert Goldstein, Second edition, Addison-Wesley. At an undergraduate level, see, for example, *Classical Dynamics* by Jerry B. Marion, Second edition, Academic Press.

and Newton's equations are transcribed as

$$-\frac{\partial V}{\partial x_i} = \frac{d}{dt} \frac{\partial T}{\partial \dot{x}_i}.$$

In rectangular coordinates (and only in rectangular coordinates)

$$\frac{\partial T}{\partial x_i} = 0$$

and — this is an important point — for a conservative system

$$\frac{\partial V}{\partial \dot{x}_i} = 0.$$

Newton's equations can then be written as

$$\frac{\partial T}{\partial x_i} - \frac{\partial V}{\partial x_i} = \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_i} - \frac{\partial V}{\partial \dot{x}_i} \right)$$

which is Lagrange's equations, usually expressed as

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0$$

where $L = T - V$ is the all-important Lagrangian function. The momentum p can be defined in terms of the Lagrangian function as

$$p_i = \frac{\partial L}{\partial \dot{x}_i}.$$

In terms of the generalized coordinates, denoted by q_i , that involve angular coordinates in addition to rectangular coordinates, the derivation of Lagrange's equations is slightly more involved. The terms $\partial T / \partial q_i$ are not zero, as in the case of rectangular coordinates, but are fictitious forces that appear because of the curvature of generalized coordinates. For example, in plane polar coordinates, where $T = (m/2)(\dot{r}^2 + r^2\dot{\theta}^2)$, we have $\partial T / \partial r = m r \dot{\theta}^2$, the centrifugal force. Lagrange's equation in terms of generalized coordinates means in

the same form, that is,²

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

with $L = T - V$ and momentum p is defined by

$$p_i = \frac{\partial L}{\partial \dot{q}_i}.$$

This definition of momentum p in terms of the Lagrangian represents a major extension of the original definition by Newton. In rectangular coordinates, it reduces to its original form, of course, but for those generalized coordinates corresponding to angles the new momentum corresponds to the angular momentum. In plane polar coordinates where $T = (m/2)(\dot{r}^2 + r^2\dot{\theta}^2)$, and $\partial V/\partial \dot{\theta} = 0$,

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

which is the angular momentum corresponding to the angular coordinate.

This new definition of momentum is technically called the canonically conjugate momentum, that is, p_i being conjugate to the generalized coordinate q_i , and this pairing of (q_i, p_i) forms the very basis of the development of quantum mechanics and, by extension, the quantum field theory. Having thus become the standard basic dynamical variables, they are simply referred to as coordinates (dropping “generalized”) and momenta (dropping “canonically conjugate”).

This new definition of the canonically conjugate momentum, or simply momentum, has far-reaching consequences when the Lagrangian formulation is adopted to the case of charged particles interacting with the electromagnetic field. Often mentioned as a supplement within the framework of classical mechanics, this casting of Maxwell’s equations into the framework of Lagrangian formulation

²Usually, Lagrange’s equations are first derived from other physical principles — D’Alembert’s principle or Hamilton’s principle — and their equivalence to Newton’s equations is shown to follow from the former. Here we follow the derivation as given in *Mechanics* by J.C. Slater and N.H. Frank, McGraw-Hill, which starts from Newton’s equation, and then derive Lagrange’s equations.

leads to non-mechanical extension of momentum and, as we will follow through in later chapters, provides the very foundation for the development of quantum electrodynamics.

Almost all forces we consider in mechanics are conservative forces, those that are functions only of positions, and certainly not functions of velocities, that is, $\partial V/\partial \dot{q}_i = 0$. There is, however, one very important case of a force that is velocity-dependent, namely, the Lorentz force on charged particles in electric and magnetic fields. In an amazing manner, the velocity-dependent Lorentz force fits perfectly into the Lagrangian formulation.

The Lagrangian equation can be written as

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = -\frac{\partial V}{\partial q_i} + \frac{d}{dt} \frac{\partial V}{\partial \dot{q}_i}.$$

For conservative systems, $\partial V/\partial \dot{q}_i = 0$. For non-conservative systems when forces, and their potentials, are velocity-dependent, it is possible to retain Lagrange's equations provided that the velocity-dependent forces are derivable from velocity-dependent potentials — also called the generalized potentials — in specific form as required by Lagrange's equations, namely, the force is derivable from its potential by the recipe, expressed back in terms of the rectangular coordinates,

$$F_i = \left(-\frac{\partial}{\partial x_i} + \frac{d}{dt} \frac{\partial}{\partial \dot{x}_i} \right) V.$$

It is a rather stringent requirement and it turns out — very fortunate for the development of quantum electrodynamics — that the Lorentz force satisfies such requirement.

Putting $c = \hbar = 1$ in the natural unit system, the Lorentz force on a charge q in electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$, is given by

$$\mathbf{F} = q\mathbf{E} + q(\mathbf{v} \times \mathbf{B})$$

where

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \quad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A}$$

and ϕ and $\mathbf{A}$ are the scalar and vector potentials, respectively, defining the four-vector potential $A_\mu = (\phi, \mathbf{A})$. After some algebra,³ the

³See Appendix 3.

Lorentz force can be expressed as

$$F_i = -\frac{\partial U}{\partial x_i} + \frac{d}{dt} \frac{\partial U}{\partial \dot{x}_i}$$

where

$$U = q\phi - q\mathbf{A} \cdot \mathbf{v}.$$

The Lagrangian for a charged particle in an electromagnetic field is thus

$$L = T - q\phi + q\mathbf{A} \cdot \mathbf{v}$$

and, as a result, the momentum — the new canonically conjugate momentum — becomes

$$\mathbf{p} = m\mathbf{v} + q\mathbf{A},$$

that is, mechanical Newtonian momentum plus an additional term involving the vector potential.

The Lagrangian formulation of mechanics was then followed by the Hamiltonian formulation based on treating the conjugate pairs of coordinates and momenta on an equal footing. This then led to the Poisson brackets for q 's and p 's and the Poisson brackets in turn led directly to quantum mechanics when they were replaced by commutators between the conjugate pairs of dynamical variables. For our purpose, we again focus on the motion of charged particles in an electromagnetic field.

In the Hamiltonian formulation, the total energy of a charged particle in an electromagnetic field is given by

$$E = \frac{1}{2m}(p_j - qA_j)(p_j - qA_j) + q\phi.$$

Comparing this expression to the total energy E for a free particle

$$E = \frac{1}{2m}p_j p_j$$

(p_j in each expression is the correct momentum for that case, that is, for free particles $m\dot{x}_j = p_j$, but for charged particles in an electromagnetic field $m\dot{x}_j = p_j - qA_j$), we arrive at the important

substitution rule: the electromagnetic interaction of charged particles is given by replacing

$$E \Rightarrow E - q\phi$$

and

$$\mathbf{p} \Rightarrow \mathbf{p} - q\mathbf{A}.$$

In relativistic notations, this substitution rule becomes a compact expression

$$p^\mu \Rightarrow p^\mu - qA^\mu$$

where

$$p^\mu = (E, \mathbf{p}) \quad \text{and} \quad A^\mu = (\phi, \mathbf{A}).$$

As we shall see later, this substitution rule, obtained when Maxwell's equations for the electromagnetic field are cast into the framework of Lagrangian and Hamiltonian formulation of mechanics, is the very foundation for the development of quantum electrodynamics and, by extension, quantum field theory. It is *that* important. One must also note that whereas the Newtonian dynamics for particles went through modifications and extensions by Lagrange and Hamilton, the equations for the electromagnetic field not only remain unmodified but also, in fact, yielded a hidden treasure of instructions on how to incorporate the electromagnetic interaction.

Canonical Quantization

Transition from classical to quantum physics, together with the discovery of relativity of space and time, represents the beginning of an epoch in the history of physics, signaling the birth of modern physics of the 20th century. Quantum physics consists, broadly, of three main theories — non-relativistic quantum mechanics, relativistic quantum mechanics, and the quantum theory of fields. In each case, the principle of quantization itself is the same and it is rooted in the canonical formalism of the Lagrangian and Hamiltonian formulation of classical mechanics. In the Hamiltonian formulation, the coordinates and momenta are accorded an equal status as independent variables to describe a dynamical system, and this is the point of departure for quantum physics.

The two most important quantities in the Hamiltonian formulation is the Hamiltonian function and the Poisson bracket. The Hamiltonian function H — or just Hamiltonian, for short — is defined by

$$H(q, p, t) = \dot{q}_i p_i - L(q, \dot{q}, t)$$

where L is the Lagrangian. In all cases that we consider, $\dot{q}_i p_i$ is equal to twice the kinetic energy, T , and with the Lagrangian being equal to

$T - V$, the Hamiltonian corresponds to the total energy of a system, namely,

$$H = T + V.$$

The Poisson bracket of two functions u, v that are functions of the canonical variables q and p is defined as

$$\{u, v\} = \frac{\partial u}{\partial q_i} \frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial v}{\partial q_i}.$$

When u and v are q 's and p 's themselves, the resulting Poisson brackets are called the fundamental Poisson brackets and they are:

$$\begin{aligned}\{q_j, q_k\} &= 0, \\ \{p_j, p_k\} &= 0,\end{aligned}$$

and

$$\{q_j, p_k\} = \delta_{jk}.$$

In terms of the Poisson brackets, the equations of motion for any functions of q and p can be expressed in a compact form. For some function u that is a function of the canonical variables and time, we have

$$\frac{du}{dt} = \{u, H\} + \frac{\partial u}{\partial t}$$

where $\{u, H\}$ is the Poisson bracket of $u(q, p, t)$ and the Hamiltonian H .

The transition from the Poisson bracket formulation of classical mechanics to the commutation relation version of quantum mechanics is affected by the formal correspondence ($\hbar$ is set equal to 1):

$$\{u, v\} \Rightarrow \frac{1}{i}[u, v]$$

where $[u, v]$ is the commutator defined by $[u, v] = uv - vu$, and on the left u, v are classical functions and on the right they are quantum mechanical operators. This transition from functions to operators and from the Poisson brackets to commutators is the very essence of quantization in a nut-shell.

In quantum mechanics, the time dependence of a system can be ascribed to either operators representing observables — momentum, energy, angular momentum and so on — or to wavefunctions representing the quantum state of a system. The former is called the Heisenberg picture and the latter Schrödinger picture (the third option is what is called the interaction picture in which both wavefunctions and operators are functions of time). In the Heisenberg picture, the equations of motion for any observable U is given by an exact counterpart of the classical equation in terms of the Poisson brackets, but with the Poisson bracket replaced by commutator, that is:

$$\frac{dU}{dt} = \frac{1}{i}[U, H] + \frac{\partial U}{\partial t}.$$

In the Schrödinger picture, operators representing observables are built up from those representing (canonically conjugate) momentum and energy by differential operators (expressed in rectangular coordinates),

$$p_j = -i\frac{\partial}{\partial x_j}$$

and since time t and $-H$ are also canonically conjugate to each other

$$E = i\frac{\partial}{\partial t}.$$

The wave equations of quantum mechanics, both non-relativistic and relativistic, are usually expressed in the Schrödinger picture and we have in the case of the non-relativistic quantum mechanics the time-dependent and time-independent Schrödinger's equations,

$$i\frac{\partial\psi(x_i, t)}{\partial t} = E\psi(x_i, t) \text{ (time-dependent)}$$

and from $E = p^2/(2m) + V$

$$\left(-\frac{1}{2m}\nabla^2 + V\right)\phi(x_i) = E\phi(x_i) \text{ (time-independent)}$$

where $\phi(x_i)$ is the space-dependent part of the total wavefunction $\psi(x_i, t)$. There are several wave equations in the relativistic quantum mechanics (Klein-Gordon, Dirac, Proca and other equations),

but they must all first and foremost satisfy the relativistic energy-momentum relations

$$E^2 = p^2 + m^2$$

from which we obtain the Klein-Gordon equation

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2 \right) \phi(x) = 0$$

or

$$(\partial_\lambda \partial^\lambda + m^2) \phi(x) = 0$$

that was mentioned in Chapter 1. As will be seen later, the quantum field theory is completely cast in the Heisenberg picture wherein the quantum mechanical wavefunctions themselves become operators.

The canonical procedure of quantization, be it non-relativistic quantum mechanics, relativistic quantum mechanics, or relativistic quantum field theory, can thus be compactly summarized as follows.

- (i) First, find the Lagrangian function L which yields the correct equations of motion via the Lagrange's equation

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0.$$

In the case of mechanical systems, $L = T - V$ and the equations of motion reduce to Newton's equations of motion.

- (ii) Define canonically conjugate momentum p with the help of L ,

$$p_i = \frac{\partial L}{\partial \dot{q}_i}.$$

- (iii) Quantization is affected when we impose the basic commutation relations

$$[q_j, q_k] = [p_j, p_k] = 0$$

and

$$[q_j, p_k] = i\delta_{jk}.$$

- (iv) The wave equations in quantum mechanics, both non-relativistic and relativistic, are obtained, in the Schrödinger picture, by the operator representation of momentum and energy (expressed in rectangular coordinates) as

$$p_j = -i \frac{\partial}{\partial x_j} \quad \text{and} \quad E = i \frac{\partial}{\partial t}.$$

As will be seen later, in the quantum field theory the very quantum mechanical wavefunctions themselves become operators for generalized coordinates and the corresponding canonically conjugate momenta are defined by the same recipe via the Lagrangian. The Lagrangian function thus is an absolutely essential element in any quantum physics, be it quantum mechanics or quantum field theory.

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4

Particles and Fields II: Duality

The departure of quantum mechanics from classical mechanics is quite drastic, rather extreme in contemporary parlance. Ordinary physical quantities are replaced by quantum mechanical operators that do not necessarily commute with each other and the Heisenberg's uncertainty principles between the canonically conjugate pairs of variables, between coordinates and momenta and between time and energy, deny the complete determinability of classical physics.

The most basic and defining characteristic of quantum mechanics — often called the central mystery of quantum mechanics — is the uniquely dual nature of matter called the wave-particle duality. In the microscopic scale of quantum world — of atoms, nuclei and elementary particles — a physical object behaves in such a way that exhibits the properties of both a wave and a particle. Often the wave-particle duality of quantum world is presented as physical objects that are *both* a wave and a particle. To us, with human intuition being nurtured in the macroscopic world, this simplistic picture of wave-particle duality is, of course, completely counterintuitive.

A quantum mechanical object is actually neither a wave in the classical sense nor a particle in the classical sense, but rather it *is*

a totally new reality, the quantum reality, that in some circumstances exhibits properties much like those of classical particles and in some other circumstances displays properties much like those of classical wave. The new quantum reality can be stated as being “neither a wave nor a particle but is something that can act sometimes much like a wave and at other times much like a particle.” The new quantum reality, the wave–particle duality, thus combines the classical dichotomy of particles and fields, waves being specific examples of a field broadly defined as an entity with spatial extension.

Let us briefly recapitulate what is meant by a particle in the classical sense. First, it has mass and occupies one geometric point in space; that is, it has no spatial extension. When it moves, under the influence of a force, it moves from one point at one time to another point at another moment in time. The entire trace of its motion is called its trajectory. Once the initial position and velocity are fixed, Newton’s equations of motion determine completely its trajectory. If the laws of motion dictate a particle to pass through a particular position A at some time t , the particle will pass through that point. There is no way the particle can be seen to be passing through any other positions at that same time. It will pass through the position A and nowhere else. Furthermore, without any force to alter its course, a particle cannot simply decide to change its direction of motion and can go to other positions. That is a no–no.

Another defining characteristic of a particle in the classical sense is the way in which it impacts, that is, how it interacts with another object. The classical particle interacts with others at a point of collision; some of its energy and momentum are transferred to others at that point of impact. It is the point-to-point transfer of energy and momentum that is the basic dynamical definition of what a particle is in the classical sense.

We can easily contradistinguish the kinematical and dynamical aspects of a classical wave from those of a classical particle. First and foremost, a wave is certainly not something that is defined at a geometrical point. On the contrary, a wave is definitely an extended object — a wave train with certain wavelength and frequency — and furthermore it does not travel along a point-to-point trajectory but

rather propagates in all directions. A sound or light wave propagates from its source in expanding spheres in all directions. In a room whose walls are shaped as the interior surface of a sphere, a wave will hit all points of the wall at the same time.

As a wave also carries its own energy and momentum, the way it propagates in all directions dictates the way it transfers energy and momentum, everywhere in all directions as it comes into contact with other matter. There is no point-to-point transfer as far as the wave is concerned. The contrast between the classical particle and wave could not have been more diagonally opposite, and this is what the new quantum reality called wave–particle duality brings together!

Now, an important caveat is in order about a matter of terminology in quantum physics. The new quantum reality, the wave–particle duality, describes a quantum *thing* that is neither a particle in classical sense nor a wave in classical sense. We can shorten the name to simply *duality*, that is, electrons, protons, neutrons and photons, etc should all be called *duality*, certainly not *particle* nor *wave*. Our reluctance or inability to part with the word “particle” is such that, however, the objects in the quantum world — be they electrons, photons, protons, neutrons, quarks and whatever — are continually referred to as “particles,” as in unstable particles, elementary particle physics and so on. What has happened is that the meaning of the word “particle” has gone through a metamorphosis: the word particle when applied to entities in the quantum world actually means duality, the wave–particle duality. Terminologies have gone from classical wave and classical particle to wave–particle duality, or just duality for short, and the “duality” has morphed back into “particle.” We will conform to this practice and from this point on in this book, the word “particle” will stand for duality and the word “particle” in classical sense will always be referred to as, “particle in the classical sense.” This is a somewhat confusing story of the evolution in the meaning of the word “particle.”

The properties of this (new quantum) particle are thus this. When it impacts, that is, interacts with other particles, it behaves much as the way a classical particle does, that is, a transfer of energy and momentum occurs at a point. But it travels in all directions like a



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classical wave. Since the impact occurs at a point, the question then arises as to what determines the particle to impact at one point at some time and at a different point at another time. In other words, what determines its preference to land at a particular point, and not elsewhere, at one time and land at another point at another time. This is the crux of the matter of quantum mechanics: the particle carries with it the information that determines the probability of its landing at a particular point.

Going back to the example of a room with the walls shaped like the interior walls of a sphere, the (quantum) particle can strike anywhere on the wall, impacting a particular point as if it were a classical particle (making a point mark on the wall). If you repeat the experiment again and again, the particle will land at points (one at a time) all over the wall, but with varying probabilities, at some points more often than at some other points.

The defining properties of the (quantum) particle can thus be summarized as:

- (i) It spreads like a classical wave, in all directions.
- (ii) However, it impacts like a classical particle.
- (iii) It carries with it its own information on the probability of where it is likely to impact.

The next logical question then is what determines its probability. And this is what the equations of quantum mechanics, such as the Schrödinger's equation of non-relativistic quantum mechanics, provide as their solutions, namely, wavefunctions.

This is phase II in the evolution of particles and fields, first the classical dichotomy and now the quantum duality.

5

Equations for Duality

The wavefunctions for a particle (in the new sense of wave–particle duality) are to be determined as solutions of quantum mechanical wave equations and these wavefunctions provide information on the probability of the particle impacting at or near a particular position. We have already mentioned the quantum mechanical differential operators corresponding to momentum and energy and by substituting these operator expressions to either the non-relativistic formula for total energy or the relativistic one, we obtain the corresponding equations of quantum mechanics.

The Schrödinger's Equation

In non-relativistic quantum mechanics, the equation in question is the Schrödinger's equation, which is the central and only wave equation for non-relativistic quantum mechanics. As mentioned in Chapter 3, the time-dependent Schrödinger's equation is

$$i\frac{\partial\psi(x_i,t)}{\partial t} = E\psi(x_i,t).$$